Partition Function of a Quadratic Functional and Semiclassical Approximation for Witten's 3-Manifold Invariant[1]

David H. Adams[2] and Siddhartha Sen

School of Mathematics, Trinity College, Dublin 2, Ireland.

email: dadams,sen@maths.tcd.ie

Abstract

An extension of the method and results of A. Schwarz for evaluating the partition function of a quadratic functional is presented. This enables the partition functions to be evaluated for a wide class of quadratic functionals of interest in topological quantum field theory, for which no method has previously been available. In particular it enables the partition functions appearing in the semiclassical approximation for the Witten-invariant to be evaluated in the most general case. The resulting $k-$dependence is precisely that conjectured by D. Freed and R. Gompf.

1

1 Introduction and summary

Partition functions of quadratic functionals (of the form (1.1) below) are important when studying the topology of manifolds via the methods of quantum field theory (QFT). In 1978 A. Schwarz [24] [25] showed that the Ray-Singer analytic torsion [20], a topological invariant, could be obtained within a QFT framework as the partition function of a certain quadratic functional. Since then it has been realised that other topological formulae and results can be obtained within a QFT framework (see [8] and [9] for reviews); this often involves evaluating partition functions of quadratic functionals.

These partition functions play an important role in recent interaction between QFT and 3-dimensional topology. In 1988 E. Witten [27] constructed a powerful new invariant of 3-manifolds within the framework of Chern-Simons gauge theory, a topological QFT[3]. Since its discovery the Witten-invariant has attracted considerable interest and has been explicitly calculated for a number of 3-manifolds and gauge groups, see for example [17],[10],[16],[23], [18] and the references therein. As well as providing a new tool for studying the topology of 3-manifolds Witten's work opens up a possibility for testing predictions of the standard methods of QFT. This is interesting not only from a physics point of view, but from a mathematics point of view as well: If the predictions of QFT hold in this case then, as we will discuss at the end of this §, it indicates deep and hitherto unexplored relationships between different areas of mathematics. The Witten-invariant associates to a 3-manifold M and gauge group G a function $Z_W(k)$ of a parameter $k \in \mathbf{Z}$, and the most basic prediction of QFT is an expression for $Z_W(k)$ in the limit $k \to \infty$, known as the semiclassical approximation. This expression, (1.14) below, is essentially a sum over partition functions of quadratic functionals.

The large$-k$ limit of $Z_W(k)$ has been explicitly calculated for a number of 3-manifolds (with $G = SU(2)$) in a program initiated by D. Freed and R. Gompf in [12] [13][4]. However, it has not been possible to make a complete comparison with the semiclassical approximation predicted by QFT because no method has existed for evaluating the partition functions appearing in

---

[3]It was the subsequent work of Reshetikin-Turaev [22] and K. Walker [26] that showed that this invariant really is a well-defined topological invariant.

[4]In this work the large$-k$ limit of $Z_W(k)$ was calculated numerically for a number of lens spaces and Brieskorn spheres. It was later calculated analytically by L. Jeffrey [16] for lens spaces and torus bundles over the circle, and by L. Rozansky [23] for Seifert manifolds.

this expression in general. In a very restricted case it has been possible to partially evaluate the partition functions, this was done by Witten in [27, §2]. The expression obtained is independent of a choice of metric used to construct it, except for a phase factor. Witten showed that the metric-dependence of the phase factor can be cancelled by adding a "geometric counterterm", i.e. functional of the metric to the phase of each partition function.

Based on the expression obtained in this restricted case and on the results of their numerical calculations of the large$-k$ limit of $Z_W(k)$, Freed and Gompf conjectured an expression for the partition functions in the semiclassical approximation in the general case, which leads to agreement with the large$-k$ limit of $Z_W(k)$ [5] In particular they conjectured the form of the dependence of the partition functions on the parameter k; this is not obtained in Witten's partial evaluation of the partition functions in the restricted case.

In this paper we give a method for evaluating the partition function for a wide class of quadratic functionals for which no method has previously existed. This includes all the partition functions appearing in the semiclassical approximation for the Witten-invariant. The expression we obtain in this case agrees with that conjectured by Freed and Gompf except for a discrepancy in the phase and overall numerical factor. In particular the conjectured $k-$dependence is obtained. The phase differs from the conjectured expression by Witten's geometric counterterm; as before this must be added by hand to obtain a topological invariant which agrees with the large$-k$ limit of $Z_W(k)$. We discuss how the discrepancy in the numerical factor can be understood from recent work by the first author combined with the results of Rozansky [23]. Thus we resolve a number of the "deeper theoretical mysteries" posed as problems for the reader by Freed and Gompf in the conclusion of their paper [12].

Our method for evaluating partition functions of quadratic functionals is an extension and refinement of the method of A. Schwarz [24] [25]. We prove that certain invariance-properties of the partition function derived by Schwarz in a restricted case continue to hold in the general case which we consider. This leads to our expression for the partition function being a topological invariant for a wider class of quadratic functionals. For example

---

[5]The agreement is modulo a discrepancy in the overall numerical factor. This discrepancy was removed in the subsequent works [16] and [23] where the conjectured expression of Freed and Gompf was refined.

the Ray-Singer torsion "as a function of the cohomology" [21] is obtained as the partition function of a quadratic functional. We also derive a new invariance-property of the partition function which relates the previous ones. We derive expressions for the phase of the partition function and the dependence of the partition function on a complex-valued scaling parameter multiplying the quadratic functional. This was not considered in Schwarz's work[6] and is necessary for evaluating the partition functions in the semiclassical approximation for the Witten-invariant.

We apply our metric-independence results to derive a result concerning the usual Ray-Singer torsion of a flat connection. When the cohomology of the connection is non-vanishing the torsion is metric-dependent, however we show that in certain cases the metric-dependence factors out in a simple way as a power of the volume of the manifold to give a topological invariant.

To motivate the method (and also to demonstrate it in a simple context) we devote a section of this paper (§2) to studying the partition function of a quadratic functional on a finite-dimensional vectorspace. In this case it turns out that the method can be motivated by requiring that a symmetry-property of the partition function of a non-degenerate quadratic functional continues to hold when the functional is degenerate.

The results we obtain in this finite-dimensional case may be of independent interest in algebraic geometry. It turns out that the partition function is essentially the determinant of a complex of linear maps between vectorspaces. Given inner products in these vectorspaces, and in the cohomology spaces of the complex, the norm of the determinant is a well-defined number. We derive formulae expressing the change in (the norm of) the determinant under changes in the inner products and changes in the maps between the vectorspaces. These formulae hold for an arbitrary (finite) complex of linear maps and are generalisations of formulae for ordinary determinants. They are simple to write down but non-trivial to prove. From them we find classes of changes under which the determinant is invariant.

We now briefly describe what the method involves (the details will be given in §2 and §3); this will enable us to describe our results more concretely. The partition functions of quadratic functionals which we consider are formal integrals over spaces of functions, or more generally sections in a vectorbundle

---

[6]The expression derived by Schwarz for the partition function is really the modulus of the partition function.

4

over a manifold M. They have the form

$$Z(\beta) = \int_\Gamma \mathcal{D}\omega \, e^{-\beta S(\omega)} \tag{1.1}$$

where $S(\omega)$ is a real-valued quadratic functional on the space Γ of sections ω (i.e. $S(\omega) = F(\omega, \omega)$ where $F(\omega, v)$ is a real-valued bilinear functional on Γ) and β is a complex-valued scaling parameter. (Typically β is either real or purely imaginary; sometimes β is taken to be a constant equal to 1 or $-i$). The expression (1.1) is mathematically ill-defined in general. This is due firstly to the fact that the integration is over an infinite-dimensional vectorspace Γ (when $dim M > 0$), and secondly, even when Γ is finite-dimensional (i.e. when $dim M = 0$) (1.1) diverges unless $\beta S(\omega)$ is positive and non-degenerate. The method we describe for evaluating and normalising (1.1) involves formal manipulations with mathematically ill-defined quantities. It leads to a well-defined finite expression for $Z(\beta)$ in the cases with which we will be concerned, and the results we subsequently derive for this expression are all mathematically rigorous.

A choice of inner product $\langle \cdot, \cdot \rangle$ in Γ determines (formally) an integration measure $\mathcal{D}\omega$ and allows us to write $S(\omega) = \langle \omega, T\omega \rangle$ where T is a uniquely determined selfadjoint map. Decomposing $\Gamma = ker(T) \oplus ker(T)^\perp$, $\omega = (\omega_1, \omega_2)$ and $\mathcal{D}\omega = \mathcal{D}\omega_1 \mathcal{D}\omega_2$ we can formally write

$$\begin{aligned}
Z(\beta) &= \int_{ker(T)} \mathcal{D}\omega_1 \left(\int_{ker(T)^\perp} \mathcal{D}\omega_2 \, e^{-\beta \langle (\omega_1, \omega_2), T(\omega_1, \omega_2) \rangle} \right) \\
&= V(ker(T)) \int_{ker(T)^\perp} \mathcal{D}\omega_2 \, e^{-\langle \omega_2, \beta \widetilde{T} \omega_2 \rangle} \tag{1.2}
\end{aligned}$$

where $\widetilde{T} \colon ker(T)^\perp \xrightarrow{\simeq} ker(T)^\perp$ is the restriction of T to $ker(T)^\perp$ and $V(ker(T))$ is the (divergent) volume of $ker(T)$. For simplicity we assume for the moment that $\beta \in \mathbf{R}_+$ and $S(\omega)$ is positive. Then (1.2) can be formally evaluated to get

$$Z(\beta) = \pi^{\zeta/2} \, \beta^{-\zeta/2} \, det(\widetilde{T})^{-1/2} \, V(ker(T)) \tag{1.3}$$

where $\zeta = dim\Gamma - dim(ker(T))$ [7]. All the factors in the formal expression (1.3) are divergent in general; however in the cases with which we will be concerned ζ and $det(\widetilde{T})$ can be regularised via zeta-function regularisation techniques to obtain finite expressions. Then, if S is non-degenerate, $ker(T) = 0$

[7] The numerical factor $\pi^{\zeta/2}$ in (1.3) should not be discarded because it is relevant when comparing the semiclassical approximation with the large$-k$ limit of the Witten-invariant.

and (1.3) gives a finite expression for the partition function, which depends on the choice of inner product $\langle \cdot, \cdot \rangle$ in Γ. When $S(\omega)$ is degenerate the partition function diverges due to the divergent volume $V(ker(T))$ in (1.3) and must be "normalised" to obtain a finite expression. The simplest way to do this is to simply divide out (i.e. discard) the factor $V(ker(T))$ in (1.3). However, this is not compatible with the usual QFT procedure of Faddeev and Popov. Also, as we will see in §2, it does not preserve a symmetry-property of the partition function of non-degenerate functionals under changes in the inner product $\langle \cdot, \cdot \rangle$ in Γ. The method for evaluating the partition function in the degenerate case requires the functional S to have an additional structure associated with it, namely a resolvent. A resolvent $R(S)$ of S is a chain of linear maps

$$0 \longrightarrow \Gamma_N \xrightarrow{T_N} \Gamma_{N-1} \xrightarrow{T_{N-1}} \ldots \longrightarrow \Gamma_1 \xrightarrow{T_1} ker(T) \longrightarrow 0 \qquad (1.4)$$

with the property $T_k T_{k+1} = 0$ for all $k = 0, 1, \ldots, N$. It determines cohomology spaces

$$H^k(R(S)) = ker(T_k) \big/ Im(T_{k+1}) \qquad (1.5)$$

for all $k = 0, 1, \ldots, N$. The method given by Schwarz [25] requires the cohomology of the resolvent to vanish. In this case, given an inner product $\langle \cdot, \cdot \rangle_k$ in each space Γ_k, the volume $V(ker(T))$ in (1.3) can be formally evaluated from the resolvent (1.4) in terms of the divergent volumes $V(\Gamma_k)$. (This can be considered as a generalisation of the Faddeev-Popov method; the details will be given in §2). An expression for the partition function is then obtained by substituting the expression for $V(ker(T))$ into (1.3). It is normalised by dividing out the divergent volumes $V(\Gamma_k)$. The resulting expression is

$$Z(\beta) = \pi^{\zeta/2} \beta^{-\zeta/2} det(\widetilde{T})^{-1/2} \prod_{k=1}^{N} det(\widetilde{T}_k^* \widetilde{T}_k)^{\frac{1}{2}(-1)^{k-1}} \qquad (1.6)$$

where $\widetilde{T}_k : ker(T_k)^\perp \xrightarrow{\simeq} Im(T_k)$ is the restriction of T_k to $ker(T_k)^\perp$. (This expression, without the factors $\pi^{\zeta/2}$ and $\beta^{-\zeta/2}$, is the one obtained by Schwarz in [25]). When the resolvent is elliptic (defined in §3) the determinants in (1.6) can be regularised by standard zeta-function regularisation techniques as shown in [25]. As we will show, the quantity ζ in (1.6) can also be regularised by zeta-regularisation when the resolvent is elliptic. Thus a finite expression for the partition function (1.6) is obtained. It depends on the

6

choice of resolvent (1.4) and inner products $\langle \cdot, \cdot \rangle_k$ in the Γ_k. We will show in §2 that in the finite-dimensional case (1.6) has a symmetry-property under changes in the inner products $\langle \cdot, \cdot \rangle_k$ which generalises a symmetry-property of partition functions of non-degenerate functionals.

The main examples of quadratic functionals of interest from a topological point of view (including those appearing in the semiclassical approximation for the Witten-invariant) have canonical elliptic resolvents associated with them. However, the cohomology of these resolvents is non-vanishing in general. Hence the need to extend the method to the case where the cohomology of the resolvent is non-vanishing (since no other methods exist for dealing with this case). We do this in this paper. Our extension of the method involves choosing inner products in the cohomology spaces $H^k(R(S))$ and uses the maps

$$\Phi_k : \mathcal{H}_k \xrightarrow{\simeq} H^k(R(S)) \tag{1.7}$$

obtained as the restriction of the projection maps $ker(T_k) \to H^k(R(S))$ to $\mathcal{H}_k = Im(T_{k+1})^{\perp} \cap ker(T_k)$. (The details are given in §2). With a choice of orientation for each space $H^k(R(S))$ the dependence of the partition function on the inner products in the $H^k(R(S))$ enters through the volume forms determined by the orientations and inner products. Considered as a functional of these volume forms the partition function can be interpreted as an element

$$Z(\beta) \in \otimes_{k=0}^{N} \Lambda^{max} H^k(R(S))^{*^{k+1}}. \tag{1.8}$$

(Here W^* denotes the dual of a vectorspace W and W^{*^k} is identified with W or W^* for k even or odd respectively).

In the general case where $\beta \in \mathbf{C}$ and the quadratic functional S is not required to be positive we formally evaluate the integral (1.2) in §2 and obtain a finite expression via analytic continuation in β. The resulting expression for the partition function $Z(\beta)$ involves a phase factor; this is well-defined for $\beta \in \mathbf{C} - \mathbf{R}$; for $\beta \in \mathbf{R}$ there is an ambiguity analogous to the ambiguity in $\sqrt{-1} = \pm i$. (The case where $\beta = i\lambda$ is purely imaginary is relevant for the semiclassical approximation for the Witten-invariant). The following expression is obtained: For $\beta = |\beta| e^{i\theta} \in \mathbf{C}_{\pm}$ with $\theta \in [-\pi, \pi]$ we get

$$\begin{aligned} Z(\beta) \;=\; & \pi^{\zeta/2} e^{-\frac{i\pi}{4}((\frac{2\theta}{\pi} \mp 1)\zeta \pm \eta)} |\beta|^{-\zeta/2} \\ & \times det(\widetilde{T}^2)^{-1/4} \prod_{k=0}^{N} \left(det(\widetilde{T}_{k+1}^* \widetilde{T}_{k+1}) det(\Phi_k^* \Phi_k)^{-1} \right)^{\frac{1}{2}(-1)^k} \end{aligned} \tag{1.9}$$

In particular, for $\beta = i\lambda$, $\lambda > 0$,

$$Z(\beta) = \pi^{\zeta/2}\, e^{-\frac{i\pi}{4}\eta}\, \lambda^{-\zeta/2}\, det(\widetilde{T}^2)^{-1/4} \prod_{k=0}^{N} \Big(det(\widetilde{T}^*_{k+1}\widetilde{T}_{k+1}) det(\Phi^*_k\Phi_k)^{-1} \Big)^{\frac{1}{2}(-1)^k}.$$

(1.10)

When the resolvent (1.4) is elliptic the determinants $det(\widetilde{T}^2)$ and $det(\widetilde{T}^*_k\widetilde{T}_k)$ are given well-defined finite values via zeta-function regularisation, and the maps Φ_k (given by (1.7)) are finite-dimensional. The quantities ζ and η are formally given by $\zeta(0\,|\,|T|)$ and $\eta(0\,|\,T)$, where $\zeta(s\,|\,|T|)$ and $\eta(s\,|\,T)$ are the zeta- and eta-functions of $|T|$ and T respectively (with $|T| = \sqrt{T^2}$ defined via spectral theory). These are given well-defined finite values via analytic continuation: We show (theorems 3.2 and 3.3) that these functions are regular at $s = 0$, and that when $dimM$ is odd

$$\zeta = \zeta(0\,|\,|T|) = \sum_{k=0}^{N}(-1)^{k+1}dimH^k(R(S)).$$

(1.11)

Thus a well-defined expression for the partition function (1.9) is obtained (up to a phase ambiguity for $\beta \in \mathbf{R}$). It depends on the choice of elliptic resolvent $R(S)$ for S and on choices of inner products in the spaces Γ_k and in the cohomology spaces $H^k(R(S))$.

For elliptic resolvents the spaces Γ_k are the spaces of smooth sections in vectorbundles over the manifold M, and the inner products in the Γ_k are constructed from Hermitian structures in the bundles and a metric on M. In [25] in the restricted case where the cohomology of the resolvent vanishes and where β is a constant equal to 1 Schwarz derived formulae for the variation of (the modulus of) the partition function under variation of the inner products in the Γ_k (theorem 1$'$ in [25]) and under a certain variation of the maps T_k in the resolvent (theorem 2$'$ in [25]). From these it followed that when the (compact, closed) manifold M has odd dimension the partition function is invariant under these variations. In particular, when the functional S and resolvent $R(S)$ are topological (i.e. their definitions do not require choices of Hermitian structures or metric on M) the partition function is a topological invariant. We show in §3 that these results continue to hold in the general case, i.e. for non-vanishing cohomology and arbitrary $\beta \in \mathbf{C}$ [8]. In addition

---

[8]The metric-independence results are for the modulus of the partition function; the phase is metric-dependent in general

we derive a new invariance-property of the partition function under certain simultaneous changes in the inner products and in the maps T_k. The formula for the change in the partition function in this case is unlike the previous ones in that its derivation involves only linear algebra as opposed to elliptic operator theory (heat kernel expansion), and holds for finite changes, not just infinitessimal changes. It relates the other formulae (i.e. the generalisations of theorems $1'$ and $2'$ of Schwarz [25]) in that by combining it with the first of these formulae we immediately obtain the second.

Finally we describe how the method enables the partition functions in the semiclassical approximation for the Witten-invariant to be evaluated. The Witten-invariant is formally given as the partition function of the Chern-Simons gauge theory:

$$Z_W(k) = \int \mathcal{D}A \, e^{ikCS(A)} \qquad k \in \mathbf{Z} \tag{1.12}$$

where the Chern-Simons functional is

$$CS(A) = \frac{1}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{1.13}$$

The formal integration in (1.12) is over the gauge fields A in a trivial bundle, i.e. 1-forms on the 3-manifold M with values in the Lie algebra $\mathbf{g}$ of the gauge group G (a compact Lie group). (The expression (1.13) is for the case where $G = SU(N)$, identified with its fundamental representation. For the correct expression in the general case, and more information on the Chern-Simons functional, see [11].) In the limit of large k QFT predicts that (1.12) is given by its semiclassical approximation [27, §2]:

$$\sum_{[A_f]} e^{ikCS(A_f)} \int \mathcal{D}\omega \, e^{\frac{ik}{4\pi} \int_M Tr(\omega \wedge d_{A_f}\omega)} . \tag{1.14}$$

The sum is over representatives A_f for each point $[A_f]$ in the moduli-space of flat gauge fields on M. The ω are Lie-algebra-valued 1-forms and d_{A_f} is the covariant derivative determined by A_f, i.e. $d_{A_f}\omega = d\omega + [A_f, \omega]$. In (1.14) we are assuming that the moduli-space is discrete, i.e. $H^1(d_{A_f}) = 0$ for all flat gauge fields A_f. (When the moduli-space is not discrete then in certain cases the sum in (1.14) can be replaced by an integral of a form of top degree on the moduli-space, as pointed out by L. Jeffrey [16, §5]). The summand

in (1.14) can be shown to be independent of the choice of representative A_f for each gauge-equivalence class.

The formal integrals in the semiclassical approximation (1.14) are partition functions of quadratic functionals. They were partially evaluated by Witten in [27, §2] in the restricted case where the cohomology of d_{A_f} vanishes. Wittens method (which used gauge-fixing implemented via a Lagrange-multiplier field) involves a "rescaling" of the fields ω in (1.14) equivalent to setting $\frac{k}{4\pi} = 1$, so the $k-$dependence of the partition functions is not obtained[9].

The method we describe in this paper enables the partition functions in (1.14) to be evaluated in complete generality, i.e. the cohomology of d_{A_f} is not required to vanish and the dependence on the parameter k is explicitly determined. Each partition function in (1.14) is of the form (1.1) with

$$S(\omega) = \int_M -\lambda_{\mathbf{g}} Tr(\omega \wedge d_{A_f}\omega) \quad , \qquad \beta = \frac{ik}{4\pi\lambda_{\mathbf{g}}} \tag{1.15}$$

where $\lambda_{\mathbf{g}} \in \mathbf{R}_+$ is an arbitrary parameter. There is a natural choice of inner products in the spaces $\Omega^q(M, \mathbf{g})$ of $\mathbf{g}$-valued q-forms in terms of which we have $S(\omega) = \langle \omega, *d_{A_f(1)}\omega \rangle_{\lambda_{\mathbf{g}}}$. The quadratic functional $S(\omega)$ in (1.15) has the canonical elliptic resolvent

$$0 \longrightarrow \Omega^0(M, \mathbf{g}) \xrightarrow{d_{A_f(0)}} ker(d_{A_f(1)}) \longrightarrow 0 \tag{1.16}$$

The partition function is then given by (1.10) with $\lambda = \frac{k}{4\pi\lambda_{\mathbf{g}}}$ and $T = *d_{A_f(1)}$. In particular, since the resolvent (1.16) has cohomology spaces $H^0(R(S)) = H^1(d_{A_f})$ and $H^1(R(S)) = H^0(d_{A_f})$ the $k-$dependence of the partition function is given by (1.10) and (1.11) to be

$$\sim k^{-\frac{1}{2}(dim H^0(d_{A_f}) - dim H^1(d_{A_f}))} \tag{1.17}$$

This is precisely the $k-$dependence conjectured by Freed and Gompf in [12], and as we show in §4 the expression for the partition function obtained from (1.10) agrees with their conjectured expression up to an overall numerical factor, except for Witten's geometric counterterms in the phase.

---

[9]Actually the method we give also leads to the partition function being independent of k when the cohomology of d_{A_f} vanishes.

The discrepancy in the numerical factor between the expression we obtain and that conjectured in [12] is due in part to the factor $\pi^{\zeta/2}$ in (1.3), which was not taken into account in [12], but this does not explain it completely. The discrepancy can be understood from work by the first author [1] combined with results of Rozansky [23]. We will discuss this in §4 and illustrate it with an explicit calculation for $M = S^3$.

We mentioned previously that testing the prediction of QFT described above is interesting from a mathematics point of view. The reason, as pointed out in [12] [13], is as follows. The method and mathematical machinery used to evaluate the Witten-invariant $Z_W(k)$ is very different from that used to derive its semiclassical approximation. Evaluation of the Witten-invariant is based on the axioms of topological QFT [3] [4] and exploits a connection between Chern-Simons gauge theory in 3 dimensions and 2-dimensional conformal field theory. It draws on topology (surgery techniques) and algebra (representation theory for Kac-Moody algebras). On the other hand, deriving the semiclassical approximation for the Witten-invariant by standard QFT methods, and evaluating it, draws on differential geometry (gauge theory, Hodge theory) and analysis (analytic continuation of functions). It is remarkable that through QFT we obtain a link between these different areas of mathematics.

This paper is organised as follows: In §2 we study the partition function (1.1) in the case where Γ is finite-dimensional (i.e. when the dimension of the manifold is zero). This allows us to describe the method for evaluating the partition function in a simple context. Formulae for the change in the partition function under changes in the inner products and maps in the resolvent are derived. We show that the method can be motivated by requiring that a symmetry-property of the partition function of non-degenerate quadratic functionals continues to hold in the degenerate case. In §3 the method is described for the case where Γ is infinite-dimensional and the resolvent is elliptic. Formulae for the variation of the partition function under certain variations in the structures used to construct it are derived (generalising results from §2) and resulting symmetry-properties are pointed out. These generalise results of Schwarz [25] and also lead to a new symmetry-property of the partition function (theorem 3.10). Finally, in §4 we apply the method and results of §3 to evaluate the partition functions of a particular class of topological quadratic functionals. The resulting expression involves a version of the Ray-Singer torsion as a "function of the cohomology" [21,

§3]. As a byproduct we show that the metric-dependence of the usual Ray-Singer torsion factors out as a power of the volume of the manifold in certain cases where the cohomology is non-vanishing (theorem 4.1). The partition functions evaluated in §4 include those appearing in the semiclassical approximation for the Witten-invariant, and we show that the expression obtained for these agrees with that conjectured by Freed and Gompf [12] (modulo Witten's geometric counterterms in the phase, and a discrepancy in the numerical factor which we explain). We explicitly calculate the semiclassical approximation for the case of the 3-sphere and show that it coincides with the Witten-invariant in the limit of large k.

2 The partition function in the finite-dimensional case

2.1 The partition function of a positive quadratic functional

We consider a quadratic functional $S(\omega)$ on a real vectorspace[10] of finite dimension d, set $\beta = 1$ and assume that $S \geq 0$. (We postpone the general case where $\beta \in \mathbf{C}$ and S is an arbitrary quadratic functional to the next subsection). An inner product $\langle \cdot, \cdot \rangle_0$ in Γ determines a measure $\mathcal{D}\omega$ on Γ and allows us to uniquely write $S(\omega) = \langle \omega, T\omega \rangle_0$ with $T \geq 0$ selfadjoint. If S is non-degenerate, i.e. if $S(\omega) > 0$ for all $\omega \neq 0$ then the partition function (1.1) of S is well-defined and equals

$$Z(S, \langle \cdot, \cdot \rangle_0) = \pi^{d/2} \, det(T)^{-1/2} \tag{2.1}$$

This expression has the following symmetry-property. For each $A \in GL(\Gamma)$ we obtain a new inner product $\langle \cdot, \cdot \rangle_0^A$ in Γ from $\langle \cdot, \cdot \rangle_0$ by setting

$$\langle v, \omega \rangle_0^A = \langle Av, A\omega \rangle_0 = \langle v, A^*A\omega \rangle_0 \,. \tag{2.2}$$

This determines a right group action of $GL(\Gamma)$ on the inner products in Γ; note that any inner product can be obtained from an arbitrary inner product $\langle \cdot, \cdot \rangle_0$ in this way. Under the action of $A \in GL(\Gamma)$ the map T is changed to T_A, where

$$S(\omega) = \langle \omega, T_A\omega \rangle_0^A = \langle \omega, A^*AT_A\omega \rangle_0 \;\; ; \quad T_A = (A^*A)^{-1}T \,. \tag{2.3}$$

---

[10]The method and results described in the following also apply for real-valued quadratic functionals on complex vectorspaces, see the remark at the end of this §.

Note that T_A is selfadjoint w.r.t. $\langle\,\cdot\,,\cdot\,\rangle_0^A$. It follows from (2.1) that

$$Z(S, \langle\,\cdot\,,\cdot\,\rangle_0^A) = |det(A)|\, Z(S, \langle\,\cdot\,,\cdot\,\rangle_0)\,. \tag{2.4}$$

Thus we see that for non-degenerate S the partition function (2.1) is invariant under the group action of $SL|\Gamma|$ on $\langle\,\cdot\,,\cdot\,\rangle_0$, where $SL|\Gamma|$ denotes the subgroup of $GL(\Gamma)$ consisting of the A with $|det(A)| = 1$.

If the functional S is degenerate then $ker(T){\neq}0$ and the partition function is given by the expression (1.3), which diverges due to the divergent volume factor $V(ker(T))$. To obtain a finite expression in this case the partition function must be normalised. The simplest way to do this is to divide out (i.e. discard) the factor $V(ker(T))$ in (1.3); this gives $Z = \pi^{\zeta/2}\, det(\widetilde{T})^{-1/2}$. However, this expression does not have the invariance-property under the action of $SL|\Gamma|$ on $\langle\,\cdot\,,\cdot\,\rangle_0$ described above. We can preserve this invariance-property by proceeding as follows. Choose a resolvent $R(S)$ for S as in (1.4), i.e. a chain of linear maps between real finite-dimensional vectorspaces:

$$0 \longrightarrow \Gamma_N \xrightarrow{T_N} \Gamma_{N-1} \xrightarrow{T_{N-1}} \ldots \longrightarrow \Gamma_1 \xrightarrow{T_1} ker(S) \longrightarrow 0 \tag{2.5}$$

with the property that $T_k T_{k+1} = 0$ for all $k = 0, 1, \ldots, N$. (Note that $ker(S) = S^{-1}(0)$ is the same as $ker(T)$ since we are assuming $S{\geq}0$.) We assume to begin with that the cohomology spaces (1.5) of the resolvent all vanish. Then, choosing an inner product $\langle\,\cdot\,,\cdot\,\rangle_k$ in each Γ_k, the resolvent (2.5) enables us to formally calculate $V(ker(S))$ in terms of the divergent volumes $V(\Gamma_k)$: The maps T_k restrict to maps $\widetilde{T}_k \colon ker(T_k)^{\perp} \xrightarrow{\cong} Im(T_k)$, which leads to the formal relation

$$V(Im(T_k)) = |det(\widetilde{T}_k)|\, V(ker(T_k)^{\perp}) = det(\widetilde{T}_k^* \widetilde{T}_k)^{1/2}\, V(ker(T_k)^{\perp})\,. \tag{2.6}$$

It is possible that $\widetilde{T}_k = 0$ and we define the determinant of the zero-map on the zero-dimensional vectorspace to be equal to 1 here and in the following. From the orthogonal decomposition $\Gamma_k = ker(T_k) \oplus ker(T_k)^{\perp}$ we get the formal relation

$$V(\Gamma_k) = V(ker(T_k)) \cdot V(ker(T_k)^{\perp})\,. \tag{2.7}$$

Combining this with (2.6) gives

$$V(Im(T_k)) = det(\widetilde{T}_k^* \widetilde{T}_k)^{1/2}\, V(\Gamma_k)\, V(ker(T_k))^{-1}\,. \tag{2.8}$$

13

The assumption of vanishing cohomology means $ker(T_k) = Im(T_{k+1})$, so from (2.8)

$$V(ker(T_k)) = det(\widetilde{T}_{k+1}^* \widetilde{T}_{k+1})^{1/2} \, V(\Gamma_{k+1}) \, V(ker(T_{k+1}))^{-1} \, . \qquad (2.9)$$

Now a simple induction argument based on (2.9) gives

$$V(ker(S)) = \prod_{k=1}^{N} \left(det(\widetilde{T}_k^* \widetilde{T}_k)^{1/2} \, V(\Gamma_k) \right)^{(-1)^{k-1}} \, . \qquad (2.10)$$

Substituting this for $V(ker(T))$ in (1.3) gives a formal expression for the partition function. We normalise this expression by dividing out the divergent volumes $V(\Gamma_k)$. This gives

$$Z(R(S), \langle \cdot , \cdot \rangle) = \pi^{\zeta/2} \, det(\widetilde{T})^{-1/2} \, \prod_{k=1}^{N} det(\widetilde{T}_k^* \widetilde{T}_k)^{\frac{1}{2}(-1)^{k-1}} \, . \qquad (2.11)$$

This procedure for evaluating and normalising the partition function (1.1) to obtain (2.11) is essentially (the finite-dimensional version of) Schwarz's method [25]. The partition function (2.11) depends on the choice $R(S)$ of resolvent (2.5) and choice of inner products $\langle \cdot , \cdot \rangle_k$ in the spaces Γ_k, which we collectively denote by $\langle \cdot , \cdot \rangle$.

The group action (2.2) of $GL(\Gamma)$ on $\langle \cdot , \cdot \rangle_0$ generalises to a right group action of $GL(\Gamma_N) \times \cdots \times GL(\Gamma_1) \times GL(\Gamma)$ on $\langle \cdot , \cdot \rangle$: An element $A = (A_N, \ldots, A_0)$ determines inner products $\langle \cdot , \cdot \rangle^A = \{\langle \cdot , \cdot \rangle_N^{A_N}, \ldots, \langle \cdot , \cdot \rangle_0^{A_0}\}$ with $\langle \cdot , \cdot \rangle_k^{A_k}$ defined by analogy with (2.2). The following theorem is a generalisation of (2.4) above; it shows that the invariance-property of the partition function of non-degenerate S generalises for the partition function obtained from the above method in the degenerate case.

Theorem 2.1. *Let $A = (A_N, \ldots, A_1, A_0) \in GL(\Gamma_N) \times \ldots \times GL(\Gamma_1) \times GL(\Gamma)$. The action of A on the inner products $\langle \cdot , \cdot \rangle$ changes the partition function (2.11) to*

$$Z(R(S), \langle \cdot , \cdot \rangle^A) = \left(\prod_{k=0}^{N} |det(A_k)|^{(-1)^k} \right) Z(R(S), \langle \cdot , \cdot \rangle) \qquad (2.12)$$

Corollary 2.2. *The partition function (2.11) is invariant under the action of $SL|\Gamma_N| \times \ldots \times SL|\Gamma_1| \times SL|\Gamma|$ on the inner products $\langle \cdot , \cdot \rangle$.*

The proof of the theorem rests on a celebrated theorem of Jacobi from 1834. We introduce the following notation: If $L\colon V \longrightarrow V$ is a linear map and P is a projection map on V then by $det(PLP)$ we mean the determinant of the restriction of PLP to the map $PLP\colon Im(P) \longrightarrow Im(P)$. For each map $T_k\colon \Gamma_k \longrightarrow \Gamma_{k-1}$ in the resolvent (2.5) define P_k and Q_k to be the orthogonal projections of Γ_k on $ker(T_k)$ and $ker(T_k)^{\perp}$ respectively. Let $T_k^{*(A)}$ denote the adjoint of T_k determined by the inner products $\langle \cdot, \cdot \rangle_k^{A_k}$ and $\langle \cdot, \cdot \rangle_{k-1}^{A_{k-1}}$. For subspace $W \subseteq \Gamma_k$ let $W^{\perp(A)}$ denote the orthogonal complement of W determined by $\langle \cdot, \cdot \rangle_k^{A_k}$.

The action of A on the inner products changes $det(\widetilde{T})$ to $det(\widetilde{T}_{A_0})$, where $\widetilde{T}_{A_0}$ is the restriction of T_{A_0} (defined as in (2.3)) to $ker(T_{A_0})^{\perp(A)}$, and changes $det(\widetilde{T}_k^* \widetilde{T}_k)$ to $det(\widetilde{T}_{k(A)}^{*(A)} \widetilde{T}_{k(A)})$, where $\widetilde{T}_{k(A)}$ is the restriction of T_k to $ker(T_k)^{\perp(A)}$. In the appendix we show that

$$det(\widetilde{T}_{A_0}) = det(Q_0 (A_0^* A_0)^{-1} Q_0)\, det(\widetilde{T}) \tag{2.13}$$

and

$$det(\widetilde{T}_{k(A)}^{*(A)} \widetilde{T}_{k(A)}) = det(Q_k (A_k^* A_k)^{-1} Q_k)\, det(P_{k-1}(A_{k-1}^* A_{k-1}) P_{k-1})\, det(\widetilde{T}_k^* \widetilde{T}_k). \tag{2.14}$$

It follows that the partition function (2.11) is changed to

$$
\begin{aligned}
&Z(R(S), \langle \cdot, \cdot \rangle^A) \\
&= \prod_{k=0}^{N} \Big(det(Q_k(A_k^* A_k)^{-1} Q_k)^{-1}\, det(P_k(A_k^* A_k) P_k) \Big)^{\frac{1}{2}(-1)^k}\, Z(R(S), \langle \cdot, \cdot \rangle).
\end{aligned}
\tag{2.15}
$$

We can obtain (2.12) from (2.15) using the following

Lemma 2.3. *Let $L\colon V \longrightarrow V$ be an invertible linear map on a finite-dimensional vectorspace, and let $P\colon V \longrightarrow V$ be a projection. Set $Q = I - P$, where I is the identity map. Then we have the formula*

$$det(PLP) = det(QL^{-1}Q)\, det(L). \tag{2.16}$$

This is presumably a classical result in the study of determinants; it is an easy consequence of the aforementioned theorem of Jacobi. This theorem can be found in e.g. [2, pp98–99]. Note that if L is strictly positive and

selfadjoint with respect to some inner product in V then PLP and $QL^{-1}Q$ are also strictly positive and selfadjoint, and therefore invertible. It follows that in this case we have

$$det(PLP)\, det(QL^{-1}Q)^{-1} = det(L)\,. \qquad (2.17)$$

Using (2.17) with $L = A_k^* A_k$ and $P = P_k$ we see that (2.15) is equal to (2.12), proving the theorem.

In order to compare with the infinite-dimensional case we give the expression for the variation of the partition function (2.11) under variation of the inner products. Let $A(t) = (A_N(t), \ldots, A_0(t))$ be a smooth curve in $GL(\Gamma_N) \times \cdots \times GL(\Gamma)$ with $A(0) = I$ (the identity), and set $B_k = \frac{d}{dt}\big|_{t=0}(A_k^*(t)A_k(t))$ for $k = 0, 1, \ldots, N$, then $\frac{d}{dt}\big|_{t=0}\langle u, v \rangle_k^{A_k(t)} = \langle u, B_k v \rangle_k$. For $t \to 0$ we have

$$(A_k^*(t)A_k(t))^{1/2} = 1 + \frac{1}{2}tB_k + O(t^2) = e^{\frac{1}{2}tB_k} + O(t^2)$$

which leads to

$$\frac{d}{dt}\bigg|_{t=0}|det(A_k(t))| = \frac{d}{dt}\bigg|_{t=0} det(A_k^*(t)A_k(t))^{1/2} = \frac{1}{2}\mathrm{Tr}(B_k)\,.$$

It now follows from (2.12) that

$$\frac{d}{dt}\bigg|_{t=0}Z(R(S), \langle \cdot, \cdot \rangle^{A(t)}) = \left(\sum_{k=0}^{N} \frac{1}{2}(-1)^k Tr(B_k)\right) Z(R(S), \langle \cdot, \cdot \rangle)\,. \qquad (2.18)$$

We now consider the case where the cohomology (1.5) of the resolvent (2.5) is non-vanishing. In this case the expression (2.11) for the partition function is no longer formally correct because we no longer have $V(ker(T_k)) = V(Im(T_{k+1}))$ in the formal calculation leading to (2.9). Note that if one simply defines the partition function by (2.11) in this case then the invariance of the partition function under the action of $SL|\Gamma_N| \times \cdots \times SL|\Gamma_1| \times SL|\Gamma|$ on the inner products no longer holds. We now describe how to construct the partition function in a way which maintains its invariance-properties when the cohomology of the resolvent is non-vanishing. In fact we will show that in this case the partition function can be interpreted as an element in $\otimes_{k=0}^{N}\Lambda^{max}H^k(R(S))^{*^{k+1}}$. (Here and in the following W^* is the dual of

a vectorspace W, and W^{*^k} is identified with W or W^* for k even or odd respectively).

Given a resolvent (2.5) and inner products $\langle\cdot,\cdot\rangle_k$ in the Γ_k we define $\mathcal{H}_k \subseteq ker(T_k)$ by the following orthogonal decomposition:

$$\ker(T_k) = Im(T_{k+1}) \oplus \mathcal{H}_k \,. \tag{2.19}$$

The projection map $ker(T_k) \longrightarrow H^k(R(S))$ restricts to an isomorphism

$$\Phi_k : \mathcal{H}_k \xrightarrow{\simeq} H^k(R(S)) \,. \tag{2.20}$$

The calculations leading to (2.10) can now be modified to formally evaluate the volume $V(ker(S))$ in the case where the cohomology of the resolvent is non-vanishing. First, from (2.19) we get the formal relation

$$V(ker(T_k)) = V(Im(T_{k+1})) \cdot V(\mathcal{H}_k) \,. \tag{2.21}$$

Next we pick an inner product $\langle\cdot,\cdot\rangle_{H^k}$ in each of the spaces $H^k(R(S))$ and get from (2.20) the formal relation

$$V(\mathcal{H}_k) = |det(\Phi_k^{-1})| \, V(H^k(R(S))) = det(\Phi_k^* \Phi_k)^{-1/2} \, V(H^k(R(S))) \,. \tag{2.22}$$

Substituting this into (2.21) a straightforward modification of the calculations leading to (2.10) gives

$$V(ker(S)) = \prod_{k=0}^{N} \left(det(\widetilde{T}_{k+1}^* \widetilde{T}_{k+1})^{1/2} det(\Phi_k^* \Phi_k)^{-1/2} \, V(\Gamma_{k+1}) V(H^k(R(S))) \right)^{(-1)^k} \tag{2.23}$$

Substituting this for $V(ker(T))$ in (1.3) gives a formal expression for the partition function. We normalise this expression by dividing out the divergent volumes $V(\Gamma_k)$ and $V(H^k(R(S)))$. This gives

$$\begin{aligned} &Z(R(S), \langle\cdot,\cdot\rangle_H, \langle\cdot,\cdot\rangle) \\ &= \pi^{\zeta/2} det(\widetilde{T})^{-1/2} \prod_{k=0}^{N} \left(det(\widetilde{T}_{k+1}^* \widetilde{T}_{k+1})^{1/2} \, det(\Phi_k^* \Phi_k)^{-1/2} \right)^{(-1)^k} \,. \end{aligned} \tag{2.24}$$

This expression depends not only on the resolvent $R(S)$ and the inner products $\langle\cdot,\cdot\rangle$ in the Γ_k, but also on the inner products $\langle\cdot,\cdot\rangle_{H^k}$ in the $H^k(R(S))$,

which we collectively denote by $\langle \cdot , \cdot \rangle_H$. The expression (2.24) is proportional to

$$\Psi(R(S), \langle \cdot , \cdot \rangle_H, \langle \cdot , \cdot \rangle) = \prod_{k=0}^{N} det(\Phi_k^* \Phi_k)^{\frac{1}{2}(-1)^{k+1}} . \tag{2.25}$$

Considered as a functional of the inner products $\langle \cdot , \cdot \rangle_{H^k}$ in the $H^k(R(S))$ we can interpret Ψ as an element $\widehat{\Psi}$ in $\otimes_{k=0}^{N} \Lambda^{max} H^k(R(S))^{*k+1}$: The map Φ_k in (2.20) induces maps

$$\begin{aligned} \hat{\Phi}_k &: \Lambda^{max} \mathcal{H}_k \xrightarrow{\simeq} \Lambda^{max} H^k(R(S)) \\ (\hat{\Phi}_k^*)^{-1} &: \Lambda^{max} \mathcal{H}_k^* \xrightarrow{\simeq} \Lambda^{max} H^k(R(S))^* . \end{aligned}$$

Now fix orientations for the cohomology spaces $H^k(R(S))$, these induce orientations for the spaces $\mathcal{H}_k$ via the maps Φ_k which together with the inner products $\langle \cdot , \cdot \rangle_k$ determine volume elements $w_k \in \Lambda^{max} \mathcal{H}_k$ and $w_k^* \in \Lambda^{max} \mathcal{H}_k^*$. We define

$$\widehat{\Psi}(R(S), \langle \cdot , \cdot \rangle) = \otimes_{k=0}^{N} (\widehat{\Phi}_k^{*k+1})^{(-1)^{k+1}} (w_k^{*k+1}) \in \otimes_{k=0}^{N} \Lambda^{max} H^k(R(S))^{*k+1} . \tag{2.26}$$

The inner product $\langle \cdot , \cdot \rangle_{H^k}$ and orientation in each $H^k(R(S))$ determine volume elements $v_k \in \Lambda^{max} H^k(R(S))$ and $v_k^* \in \Lambda^{max} H^k(R(S))^*$. Define

$$v_H = \otimes_{k=0}^{N} v_k^{*k} \in \otimes_{k=0}^{N} \Lambda^{max} H^k(R(S))^{*k} . \tag{2.27}$$

Let $< \cdot , \cdot >_k$ denote the natural pairing of $\Lambda^{max} H^k(R(S))$ and $\Lambda^{max} H^k(R(S))^*$ then

$$< \widehat{\Phi}_k(w_k), v_k^* >_k = det(\Phi_k^* \Phi_k)^{1/2} \quad , \quad < (\widehat{\Phi}_k^*)^{-1}(w_k^*), v_k >_k = det(\Phi_k^* \Phi_k)^{-1/2}$$

and it follows, with $< \cdot , \cdot > = \otimes_{k=0}^{N} < \cdot , \cdot >_k$, that

$$\Psi(R(S), \langle \cdot , \cdot \rangle_H, \langle \cdot , \cdot \rangle) = < \widehat{\Psi}(R(S), \langle \cdot , \cdot \rangle), v_H > .$$

Thus we see that Ψ in (2.25) can be identified with $\widehat{\Psi}$ in (2.26). It follows that as a functional of the inner products $\langle \cdot , \cdot \rangle_H$ in the cohomology spaces $H^*(R(S))$ the partition function can be considered as an element

$$\widehat{Z}(R(S), \langle \cdot , \cdot \rangle) \in \otimes_{k=0}^{N} \Lambda^{max} H^k(R(S))^{*k+1} \tag{2.28}$$

with
$$Z(R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) = < \hat{Z}(R(S), \langle \cdot, \cdot \rangle), v_H > . \qquad (2.29)$$

The partition function (2.24) has the same invariance-property as before under change of the inner products in the spaces Γ_k, as the following theorem shows.

Theorem 2.4. *Let $A = (A_N, \dots, A_1, A_0) \in GL(\Gamma_N) \times \cdots \times GL(\Gamma_1) \times GL(\Gamma)$. The action of A on the inner products in the spaces Γ_k changes the partition function (2.24) to*

$$Z(R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle^A) = \left(\prod_{k=0}^{N} |det(A)|^{(-1)^k} \right) Z(R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) \quad (2.30)$$

Corollary 2.5. *The partition function (2.24) is invariant under the action of $SL|\Gamma_N| \times \cdots \times SL|\Gamma_1| \times SL|\Gamma|$ on the inner products in the spaces Γ_k.*

Theorem 2.4 and corollary 2.5 also hold when the partition function (2.24) is replaced by (2.28).

To prove the theorem we show in the appendix that the action of A on $\langle \cdot, \cdot \rangle$ changes $det(\tilde{T}_k^* \tilde{T}_k) \, det(\Phi_{k-1}^* \Phi_{k-1})^{-1}$ to

$$\begin{aligned}
det(\tilde{T}_{k(A)}^{*(A)} \tilde{T}_{k(A)}) \, det(\Phi_{k-1(A)}^{*(A)} \Phi_{k-1(A)})^{-1} \\
= \; det\Big(Q_k (A_k^* A_k)^{-1} Q_k \Big) det\Big(P_{k-1}(A_{k-1}^* A_{k-1}) P_{k-1} \Big) \\
\times \, det(\tilde{T}_k^* \tilde{T}_k) \, det(\Phi_{k-1}^* \Phi_{k-1})^{-1} . \qquad (2.31)
\end{aligned}$$

The theorem now follows from (2.13) and (2.31) in the same way that theorem 2.1 followed from (2.13) and (2.14).

The variation of the partition function (2.24) under variation of the inner products in the spaces Γ_k can be derived from (2.30) in the same way that (2.18) was derived from (2.12). The result is

$$\frac{d}{dt}\bigg|_{t=0} Z(R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle^{A(t)}) = \left(\sum_{k=0}^{N} \frac{1}{2}(-1)^k Tr(B_k) \right) Z(R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle)$$

$$(2.32)$$

where $A(t)$ and B_k are as in (2.18).

In [25] the dependence of the partition function on the maps T_k in the resolvent was investigated, which we now discuss. Given a resolvent $R(S)$ as in

19

(2.5) we can construct new resolvents as follows. For any $C = (C_N, \ldots, C_1, C_0) \in GL(\Gamma_N) \times \cdots \times GL(\Gamma_1) \times GL(ker(S))$ we define maps $T_k^C = C_{k-1}^{-1} \circ T_k \circ C_k : \Gamma_k \longrightarrow \Gamma_{k-1}$ for each $k = 1, \ldots, N$. Then by replacing each T_k in (2.5) by T_k^C we obtain a new resolvent for the functional S, which we denote by $R(S)^C$. This defines a right-action of $GL(\Gamma_N) \times \cdots \times GL(\Gamma_1) \times GL(ker(S))$ on the set of resolvents for the functional S. Note that $ker(T_k^C) = C_k^{-1}(ker(T_k))$ and $Im(T_{k+1}^C) = C_k^{-1}(Im(T_{k+1}))$, it follows that C_k^{-1} on $ker(T_k)$ induces an isomorphism

$$C_k^{-1} : H^k(R(S)^C) \overset{\simeq}{\longrightarrow} H^k(R(S)) . \qquad (2.33)$$

It follows that the dimensions of the cohomology spaces of $R(S)^C$ are the same as for $R(S)$. Conversely, if $R(S)'$ is another resolvent with the same spaces Γ_k and same dimensions of the cohomology spaces as $R(S)$ then it is easy to see that $R(S)' = R(S)^C$ for some C. Given a collection $\langle \cdot, \cdot \rangle_H = \{\langle \cdot, \cdot \rangle_{H^0}, \ldots, \langle \cdot, \cdot \rangle_{H^N}\}$ of inner products in the cohomology spaces $H(R(S))$ it follows from (2.33) that $\langle \cdot, \cdot \rangle_{H^k}^{C_k}$, defined by analogy with (2.2), defines an inner product in $H^k(R(S)^C)$; we denote the collection of these by $\langle \cdot, \cdot \rangle_H^C$. The group $GL(\Gamma_N) \times \ldots \times GL(\Gamma_1) \times GL(ker(S))$ also acts on the inner products $\langle \cdot, \cdot \rangle$ in the spaces Γ_k. (We extend $C_0 \in GL(ker(S))$ to $GL(\Gamma)$ by defining C_0 to be the identity map on $ker(S)^\perp$). The partition function has the following invariance-property:

Theorem 2.6. *For all* $C = (C_N, \ldots, C_1, C_0) \in GL(\Gamma_N) \times \cdots \times GL(\Gamma_1) \times GL(ker(S))$ *the partition function (2.24) satisfies*

$$Z(R(S)^C, \langle \cdot, \cdot \rangle_H^C, \langle \cdot, \cdot \rangle^C) = Z(R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) . \qquad (2.34)$$

Proof. Let $(T_k^C)^{*(C)}$ denote the adjoint of T_k^C w.r.t. the inner products $\langle \cdot, \cdot \rangle_k^{C_k}$ and $\langle \cdot, \cdot \rangle_{k-1}^{C_{k-1}}$, and for a subspace $W \subseteq \Gamma_k$ let $W^{\perp(C)}$ denote the orthogonal complement of W w.r.t. $\langle \cdot, \cdot \rangle_k^{C_k}$. For $v \in \Gamma_{k-1}$, $w \in \Gamma_k$,

$$\langle v, T_k^C w \rangle_{k-1}^{C_{k-1}} = \langle C_k^{-1} T_k^* C_{k-1} v, w \rangle_k^{C_k}$$

so

$$(T_k^C)^{*(C)} T_k^C = C_k^{-1} T_k^* T_k C_k . \qquad (2.35)$$

Let $\widetilde{T}_k^C$ denote the restriction of T_k^C to $ker(T_k^C)^{\perp(C)} = C_k^{-1}(ker(T_k)^\perp)$, then from (2.35)

$$det((\widetilde{T}_k^C)^{*(C)} \widetilde{T}_k^C) = det(\widetilde{T}_k^* \widetilde{T}_k) . \qquad (2.36)$$

20

Since C_0 is the identity map on $ker(S)^\perp$ we see from (2.3) that $\widetilde{T}_{C_0} = \widetilde{T}$, so

$$det(\widetilde{T}_{C_0}) = det(\widetilde{T}).\tag{2.37}$$

Let $\Phi_k^C : \mathcal{H}_k^C \xrightarrow{\simeq} H^k(R(S)^C)$ denote the map constructed by analogy with (2.20) from $\langle\cdot,\cdot\rangle^C$ and $R(S)^C$, then $\mathcal{H}_k^C = C_k^{-1}(\mathcal{H}_k)$, $H^k(R(S)^C) = C_k^{-1}(H^k(R(S))$ and $\Phi_k^C = C_k^{-1}\Phi_k C_k$. By an argument similar to that leading to (2.36) we obtain

$$(\Phi_k^C)^{*(C)}\Phi_k^C = C_k^{-1}\Phi_k^*\Phi_k C_k\tag{2.38}$$

and therefore

$$det((\Phi_k^C)^{*(C)}\Phi_k^C) = det(\Phi_k^*\Phi_k).\tag{2.39}$$

Combining (2.36), (2.37) and (2.39) with the expression (2.24) gives (2.34), proving the theorem.

Combining theorem 2.6 with theorem 2.4 allows to determine the change in the partition function resulting from the group action on the pair $(R(S), \langle\cdot,\cdot\rangle_H)$: From theorem 2.6,

$$Z(R(S)^C, \langle\cdot,\cdot\rangle_H^C, \langle\cdot,\cdot\rangle) = Z(R(S), \langle\cdot,\cdot\rangle_H, \langle\cdot,\cdot\rangle^{C^{-1}})$$

and theorem 2.4 now gives

Corollary 2.7

$$Z(R(S)^C, \langle\cdot,\cdot\rangle_H^C, \langle\cdot,\cdot\rangle) = \left(\prod_{k=0}^N |det(C_k)|^{(-1)^{k+1}}\right) Z(R(S), \langle\cdot,\cdot\rangle_H, \langle\cdot,\cdot\rangle).$$

$$\tag{2.40}$$

In particular $Z(R(S), \langle\cdot,\cdot\rangle_H, \langle\cdot,\cdot\rangle)$ *is invariant under the action of* $SL|\Gamma_N| \times \cdots \times SL|\Gamma_1| \times SL|ker(S)|$ *on the pair* $(R(S), \langle\cdot,\cdot\rangle_H)$.

This allows to derive the variation of the partition function under variation of the pair $(R(S), \langle\cdot,\cdot\rangle_H)$ by the group action: Let $C(t)$ be a smooth curve in $GL(\Gamma_N) \times \cdots \times GL(\Gamma_1) \times GL(ker(S))$ with $C(0) = I$ and set $D_k = \frac{d}{dt}\big|_{t=0}(C_k^*(t)C_k(t))$. Then from (2.40),

$$\frac{d}{dt}\bigg|_{t=0} Z(R(S)^{C(t)}, \langle\cdot,\cdot\rangle_H^{C(t)}, \langle\cdot,\cdot\rangle)$$

$$= \left(\sum_{k=0}^N \frac{1}{2}(-1)^{k+1}\mathrm{Tr}(D_k)\right) Z(R(S), \langle\cdot,\cdot\rangle_H, \langle\cdot,\cdot\rangle).\tag{2.41}$$

21

The results derived in this subsection may be of independent algebraic interest. The partition function (2.24) is the square root of the norm of the determinant of the following complex of linear maps[11] :

$$0 \longrightarrow \Gamma_N \xrightarrow{T_N} \ldots \longrightarrow \Gamma_1 \xrightarrow{T_1} \Gamma \xrightarrow{T} \Gamma \xrightarrow{T_1^*} \Gamma_1 \longrightarrow \ldots \xrightarrow{T_N^*} \Gamma_N \longrightarrow 0 \,.$$

In the special case where $S = 0$ the partition function (2.28) is the determinant of the complex

$$0 \longrightarrow \Gamma_N \xrightarrow{T_N} \ldots \longrightarrow \Gamma_2 \xrightarrow{T_2} \Gamma_1 \xrightarrow{T_1} \Gamma \longrightarrow 0 \,.$$

Therefore the preceding theorems describe transformation– and invariance– properties of determinants of complexes.

Finally we note that the partition function (2.24) has encoded in it the data characterising the vectorspaces associated with the resolvent $R(S)$: The dimensions of $ker(S)$, Γ , Γ_k , $ker(T_k)$, $Im(T_k)$ and $H^k(R(S))$ can be recovered from the scaling behavior of this partition function under the scalings $T_k \to \lambda T_k$, $\Phi_k \to \mu \Phi_k$ for $k = 0, 1, \ldots, N$. Note that if we had constructed the partition function by simply discarding the divergent volume $V(ker(T))$ in (1.3) then the resulting expression would not encode the dimensions of Γ and of $ker(S) = ker(T)$.

2.2 The partition function of an arbitrary quadratic functional scaled by a complex-valued parameter

In the following we extend the method described above to evaluate the partition function (1.1) in the general case where the functional S is not required to be positive, and where β is an arbitrary complex-valued parameter. Note first that if $S(\omega) = \langle \omega, T\omega \rangle_0$ takes negative as well as positive values then we no longer have $\ker(T) = S^{-1}(0)$ as in the preceding. However, from (2.3) it is clear that $\ker(T) \subseteq \Gamma$ is independent of the choice of inner product $\langle \cdot, \cdot \rangle_0$ in Γ and by an abuse of notation we will continue to denote this space by $\ker(S)$. With this notation a resolvent of an arbitrary quadratic functional is defined as in (2.5) above.

---

[11] For a description of the determinant of a complex of linear maps see e.g. [14, appendix A].

We decompose $ker(T)^\perp = \Gamma_+ \oplus \Gamma_-$ such that $\widetilde{T}$ restricts to a strictly positive map $\widetilde{T}_+ : \Gamma_+ \longrightarrow \Gamma_+$ and a strictly negative map $\widetilde{T}_- : \Gamma_- \longrightarrow \Gamma_-$. Then the expression (1.2) for the partition function can be formally written as

$$Z(\beta) = V(ker(S)) \left(\int_{\Gamma_+} \mathcal{D}\omega_+ \, e^{-\langle \omega_+, \beta \widetilde{T}_+ \omega_+ \rangle_0} \right) \left(\int_{\Gamma_-} \mathcal{D}\omega_- \, e^{-\langle \omega_-, \beta \widetilde{T}_- \omega_- \rangle_0} \right) \quad (2.42)$$

We evaluate the integrals $\int_{\Gamma_\pm} (\cdots)$ in (2.42) as follows. The maps $\widetilde{T}_+$ and $-\widetilde{T}_-$ are strictly positive, so $\beta \widetilde{T}_\pm = (\pm\beta)(\pm\widetilde{T}_\pm)$ is strictly positive for $\beta \in \mathbf{R}_\pm$. In this case

$$\int_{\Gamma_\pm} \mathcal{D}\omega_\pm \, e^{-\langle \omega_\pm, \beta \widetilde{T}_\pm \omega_\pm \rangle_0} = \pi^{d_\pm/2} \, (\pm\beta)^{d_\pm/2} \, det(\pm\widetilde{T}_\pm)^{-1/2} \quad (2.43)$$

where $d_\pm = dim\Gamma_\pm$. We extend (2.43) to $\beta \in \mathbf{C}$ by analytic continuation; to do this we must fix a convention for defining z^a for $z \in \mathbf{C}$ and $a \in \mathbf{R}$. The natural way to do this is to write $z = |z|e^{i\theta}$ with $\theta \in [-\pi, \pi]$ and set $z^a = |z|^a e^{i\theta a}$. This is well-defined for all $a \in \mathbf{R}$ provided $z \notin \mathbf{R}_-$; if $z \in \mathbf{R}_-$, $z \neq 0$ there is a phase ambiguity. Now from (2.43) we get the following expression for (2.42):

$$Z(\beta) = \pi^{\zeta/2} \, \beta^{d_+/2} \, (-\beta)^{d_-/2} \, det(|\widetilde{T}|)^{-1/2} \, V(ker(S)) \quad (2.44)$$

where $|\widetilde{T}| = \sqrt{\widetilde{T}^2}$ is defined via spectral theory. Let $\mathbf{C}_+$ and $\mathbf{C}_-$ denote the upper and lower halfplanes of $\mathbf{C}$ respectively. We write $\beta = |\beta|e^{i\theta}$ with $\theta \in [-\pi, \pi]$; then for $\beta \in \mathbf{C}_\pm$ we have $-\beta = |\beta|e^{i(\theta \mp \pi)}$ with $\theta \mp \pi \in [-\pi, \pi]$. Using $\zeta = d_+ + d_-$, setting $\eta = d_+ - d_-$ and formally evaluating and normalising the divergent volume $V(ker(S))$ from a resolvent $R(S)$ as previously, we obtain from (2.44) the following expression for the partition function:

$$\begin{aligned} &Z(\beta \,; R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) \\ &= \pi^{\zeta/2} e^{-\frac{i\pi}{4}((\frac{2\theta}{\pi} \mp 1)\zeta \pm \eta)} |\beta|^{-\zeta/2} \, \widetilde{Z}(R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) \quad (2.45) \end{aligned}$$

where

$$\begin{aligned} &\widetilde{Z}(R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) \\ &= det(|\widetilde{T}|)^{-1/2} \prod_{k=0}^{N} \left(det(\widetilde{T}_{k+1}^* \widetilde{T}_{k+1})^{1/2} \, det(\Phi_k^* \Phi_k)^{-1/2} \right)^{(-1)^k}. \quad (2.46) \end{aligned}$$

The partition function (2.45) is well-defined for $\beta \in \mathbf{C} - \mathbf{R}$; for $\beta \in \mathbf{R}$ there is a phase ambiguity. Note that

$$\zeta = \zeta(0 \,|\, |T|) \;, \quad \eta = \eta(0 \,|\, T) \tag{2.47}$$

where $\zeta(s \,|\, |T|)$ and $\eta(s \,|\, T)$ are the zeta- and eta-functions of $|T|$ and T respectively. In the infinite-dimensional case considered in §3 (2.47) will be used to regularise the quantities ζ and η in the partition function (2.45). By analogy with (2.28) we can consider the partition function as an element $\widehat{Z}(\beta \,;\, R(S), \langle \cdot \,, \cdot \rangle)$ in $\otimes_{k=0}^{N} \Lambda^{max} H^k(R(S))^{*^{k+1}}$.

There are 2 special cases of particular interest: Setting $\beta = 1$ in (2.45) gives

$$Z(R(S), \langle \cdot \,, \cdot \rangle_H, \langle \cdot \,, \cdot \rangle) = \pi^{\zeta/2} e^{\pm \frac{i\pi}{4}(\zeta - \eta)} \widetilde{Z}(R(S), \langle \cdot \,, \cdot \rangle_H, \langle \cdot \,, \cdot \rangle) . \tag{2.48}$$

The case of relevance to the partition functions in the semiclassical approximation (1.14) is $\beta = i\lambda$, $\lambda \in \mathbf{R}$. In this case we have $\theta = \pm \pi/2$ for $\lambda \in \mathbf{R}_\pm$ and the partition function (2.45) is

$$Z(i\lambda \,;\, R(S), \langle \cdot \,, \cdot \rangle_H, \langle \cdot \,, \cdot \rangle) = \pi^{\zeta/2} \, e^{\mp \frac{i\pi}{4}\eta} \, \lambda^{-\zeta/2} \, \widetilde{Z}(R(S), \langle \cdot \,, \cdot \rangle_H, \langle \cdot \,, \cdot \rangle) . \tag{2.49}$$

Note that ζ and η depend only on the functional S, and not on the choice of inner product in Γ. Therefore the transformation– and invariance– properties of the partition function described in the preceding continue to hold for the general partition function (2.45).

Finally we point out that the methods and results of this section also apply to real-valued quadratic functionals on complex vectorspaces. (Then for any choice of complex inner product $\langle \cdot \,, \cdot \rangle$ in Γ we can write $S(\omega) = \langle \omega, T\omega \rangle$, with T selfadjoint, as before). Since the integration in (1.1) in this case is over the real vectorspace underlying Γ, which has twice the dimension of Γ, the expressions for the partition functions in this case are the squares of those above.

3 The partition function in the infinite-dimensional case

In this § we extend the construction of the partition function in §2 to the following infinite-dimensional setup and derive its variation under variation

24

of the inner products and maps in the resolvent by the group actions introduced in §2. In doing so we extend results obtained by Schwarz in [25]. It turns out that a version of theorem 2.6 from §2 continues to hold in the following infinite-dimensional case, determining a new invariance-property of the partition function. We will use some standard results in elliptic operator theory –for the proofs see e.g. [15].

3.1   Construction of the partition function

Throughout the following M denotes a compact oriented riemannian manifold without boundary. The functionals we consider are defined on the space $\Gamma = \Gamma(M, \xi)$ of smooth sections in a real Hermitian vectorbundle[12] ξ over M. Given a metric on M and a Hermitian structure in ξ, i.e. an inner product $\langle \cdot, \cdot \rangle_x$ in each fibre ξ_x smoothly varying with $x \in M$, we obtain an inner product in Γ :

$$\langle \omega, \tau \rangle_0 = \int_M \langle \omega(x), \tau(x) \rangle_x \, vol(x) \tag{3.1}$$

where vol is the volume form on M determined by the metric and orientation. Our evaluation of the partition function of a quadratic functional $S(\omega)$ on Γ from a resolvent $R(S)$ of S (as in (2.5)) requires the following conditions to be satisfied.

(i) The functional can be written as $S(\omega) = \langle \omega, T\omega \rangle_0$ where T is a formally self-adjoint differential operator of order $d > 0$ on Γ .

(ii) The resolvent $R(S)$ in (2.5) is of the following type : Each space Γ_k is the space $\Gamma_k = \Gamma(M, \xi_k)$ of smooth sections in a real Hermitian vectorbundle ξ_k over M, and each map $T_k : \Gamma_k \longrightarrow \Gamma_{k-1}$ is a differential operator of the same order d as T in (i).

(iii) The following chain of maps determined by the resolvent $R(S)$ is an elliptic complex:

$$0 \xrightarrow{0} \Gamma_N \xrightarrow{T_N} \ldots \longrightarrow \Gamma_1 \xrightarrow{T_1} \Gamma \xrightarrow{T} \Gamma \xrightarrow{T_1^*} \Gamma_1 \longrightarrow \ldots \xrightarrow{T_N^*} \Gamma_N \xrightarrow{0} 0 \,. \tag{3.2}$$

Condition (iii) is equivalent to requiring that the Laplace-operators

$$\Delta_k = T_k^* T_k + T_{k+1} T_{k+1}^* : \Gamma_k \longrightarrow \Gamma_k \tag{3.3}$$

---

[12]Complex bundles can also be treated, in the same way as complex vectorspaces were treated in §2.

are elliptic of order $2d$ for all $k = 0, 1, \ldots, N$ (with $\Gamma_0 = \Gamma$, $T_0 = T$, $T_{N+1} = 0$). The adjoint maps T_k^* in (3.2) are obtained by choosing a Hermitian structure in each ξ_k and a metric on M and defining an inner product $\langle \cdot, \cdot \rangle_k$ in each Γ_k by analogy with (3.1). Although the complex (3.2) depends on the choice of Hermitian structures and metric, it is easy to see that the ellipticity of (3.2), or lack of it, is independent of these choices. We denote the inner products $\langle \cdot, \cdot \rangle_N, \ldots, \langle \cdot, \cdot \rangle_0$ collectively by $\langle \cdot, \cdot \rangle$ as previously.

When the functional S satisfies (i) and the resolvent $R(S)$ satisfies (ii) and (iii) we say that it is an elliptic resolvent of S. In this case it follows from Hodge-theory that the cohomology spaces (1.5) of $R(S)$ are finite-dimensional. Given an elliptic resolvent $R(S)$ for S we equip the cohomology spaces $H^k(R(S))$ with inner products $\langle \cdot, \cdot \rangle_{H^k}$, then with $\beta = |\beta| e^{i\theta} \in \mathbf{C}_\pm$ the partition function of S can be formally evaluated as in §2 to get the formal expression (2.45) :

$$Z(\beta \, ; R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle)$$
$$= \; \pi^{\zeta/2} \, e^{-\frac{i\pi}{4}((\frac{2\theta}{\pi} \mp 1)\zeta \pm \eta)} \, |\beta|^{-\zeta/2} \, \widetilde{Z}(R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) \qquad (3.4)$$

with

$$\widetilde{Z}(R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) = \widetilde{Z}(R(S), \langle \cdot, \cdot \rangle) \cdot \Psi(R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) \qquad (3.5)$$

where

$$\widetilde{Z}(R(S), \langle \cdot, \cdot \rangle) = \det(\widetilde{T}^2)^{-1/4} \prod_{k=1}^{N} \det(\widetilde{T}_k^* \widetilde{T}_k)^{\frac{1}{2}(-1)^{k-1}} \qquad (3.6)$$

and

$$\Psi(R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) = \prod_{k=0}^{N} \det(\Phi_k^* \Phi_k)^{\frac{1}{2}(-1)^{k+1}} \qquad (3.7)$$

The expressions (3.5), (3.6) and (3.7) follow from (2.46). The expression (3.4) for the partition function is not well-defined a priori because it involves the determinants of operators on infinite-dimensional vectorspaces, and because the quantities ζ and η do not have well-defined finite values in the present infinite-dimensional case.

To obtain a well-defined expression for the partition function (3.4) we use zeta-function regularisation techniques, following [25]. Note that in the present case the maps Φ_k continue to map between finite-dimensional vectorspaces so Ψ in (3.7) is well-defined. Given a choice of orientations for the

26

cohomology spaces $H^k(R(S))$ then considered as a functional of $\langle \cdot, \cdot \rangle_H$ Ψ can be interpreted as an element

$$\widehat{\Psi}(R(S), \langle \cdot, \cdot \rangle) = \otimes_{k=0}^{N} \Lambda^{max} H^k(R(S))^{*k+1} \tag{3.8}$$

as previously in (2.26). The same is therefore true of the zeta-regularised partition function which we will obtain from (3.4). Remarkably, it turns out that the zeta-regularised expression for the partition function has analogous symmetry properties to those established for the finite-dimensional case in §2 under variation of the inner products and resolvent : We will derive generalisations of the formulae (2.32) and (2.41) for the variation of the partition function under variations of the inner products in the spaces Γ_k resulting from variations of the Hermitian structures and metric on M, and under variations of the resolvent of the type considered in §2. It turns out that when the dimension of the manifold is odd the modulus of the partition function is invariant under *all* the mentioned variations. This extends results of Schwarz in [25] obtained in the case where the cohomology of the resolvent vanishes.

The zeta-regularisation techniques and subsequent formulae that we derive are based on some standard results concerning the zeta-functions of the Laplacians Δ_k in (3.3) and of $T_k^* T_k$. For the sake of completeness we give a brief summary of these. Since the complex (3.2) is elliptic the Laplace-operators (3.3) are positive, selfadjoint elliptic differential operators. Let $\{\mu_j^{(k)}\}$ denote the non-zero eigenvalues of Δ_k, with $\mu_j^{(k)} \le \mu_{j+1}^{(k)}$ for all j, and each $\mu_j^{(k)}$ appearing the same number of times as its multiplicity. The zeta-function for Δ_k is

$$\zeta(s \mid \Delta_k) = \sum_j \frac{1}{(\mu_j^{(k)})^s} . \tag{3.9}$$

This is well-defined and analytic for $s \in \mathbf{C}$ with $\mathrm{Re}(s) >> 0$; it extends by analytic continuation to a meremorphic function on $\mathbf{C}$, regular at $s = 0$. From the property $T_k \circ T_{k+1} = 0$ of the resolvent it follows that $(T_k^* T_k) \circ (T_{k+1} T_{k+1}^*) = (T_{k+1} T_{k+1}^*) \circ (T_k^* T_k) = 0$, and therefore $\{\mu_j^{(k)}\}$ is the union of the non-zero eigenvalues of $T_k^* T_k$ and $T_{k+1} T_{k+1}^*$. Since $T_{k+1} T_{k+1}^*$ has the same non-zero eigenvalues as $T_{k+1}^* T_{k+1}$ (because if $T_{k+1}^* T_{k+1}(\omega) = \lambda \omega$ then $T_{k+1} T_{k+1}^*(T_{k+1} \omega) = \lambda(T_{k+1} \omega)$, etc.) it follows that

$$\{\mu_j^{(k)}\} = \{\lambda_l^{(k)}\} \cup \{\lambda_m^{(k+1)}\} \tag{3.10}$$

where $\{\lambda_l^{(k)}\}$ and $\{\lambda_m^{(k+1)}\}$ are the non-zero eigenvalues of $T_k^*T_k$ and $T_{k+1}^*T_{k+1}$ respectively. We will need the following

Lemma 3.1. *The zeta-functions $\zeta(s \,|\, T_k^*T_k)$ (defined by analogy with (3.9)) are well-defined and analytic for $\mathrm{Re}(s) \gg 0$ for all $k = 0, 1, \ldots, N$. They extend by analytic continuation to meremorphic functions on $\mathbf{C}$, which are regular at $s = 0$, and satisfy the formulae*

$$\zeta(s \,|\, \Delta_k) = \zeta(s \,|\, T_k^*T_k) + \zeta(s \,|\, T_{k+1}^*T_{k+1}) \,. \tag{3.11}$$

Proof. Since all the $\mu_j^{(k)}$ and $\lambda_l^{(k)}$ are positive it follows from (3.10) and the fact that (3.9) is well-defined for $\mathrm{Re}(s) \gg 0$ that the $\zeta(s \,|\, T_k^*T_k)$ are well-defined for $\mathrm{Re}(s) \gg 0$. It now follows from (3.10) that the formula (3.11) holds for $\mathrm{Re}(s) \gg 0$. A simple induction argument based on (3.11) with $\mathrm{Re}(s) \gg 0$ and starting with $\zeta(s \,|\, \Delta_N) = \zeta(s \,|\, T_N^*T_N)$ (obtained by setting $k = N$ in (3.11)) shows that the $\zeta(s \,|\, T_k^*T_k)$ are analytic for $\mathrm{Re}(s) \gg 0$, and that their analytic continuations to meremorphic functions on $\mathbf{C}$ are regular at $s = 0$, because the $\zeta(s \,|\, \Delta_k)$ have these properties.

The analytic continuations of the $\zeta(s \,|\, T_k^*T_k)$ can be used to regularise the determinants appearing in the partition function in the present case : Formally we have

$$\det(\widetilde{T}_k^*\widetilde{T}_k) = \prod_j \lambda_j^{(k)} = e^{-\zeta'(0 \,|\, T_k^*T_k)} \,. \tag{3.12}$$

We use the analytic continuation of $\zeta(s \,|\, T_k^*T_k)$ to give meaning to the r.h.s. of (3.12) and substitute this for $\det(\widetilde{T}_k^*\widetilde{T}_k)$ in the expression (3.6) above. From (2.47) we see that the quantities ζ and η in (3.4) are formally given by

$$\zeta = \zeta(0 \,|\, |T|) \ , \quad \eta = \eta(0 \,|\, T) \tag{3.13}$$

where $\zeta(s \,|\, |T|)$ and $\eta(s \,|\, T)$ are the zeta- and eta-functions of $|T|$ and T respectively. We could also obtain (3.13) by repeating the formal calculations in §2.2 with the determinants replaced by their zeta-regularisations (defined by analogy with (3.12)). Theorems 3.2 and 3.3 below enable us to give finite, well-defined meaning to ζ and η in (3.13) via the analytic continuations of $\zeta(s \,|\, |T|)$ and $\eta(s \,|\, T)$ to $s = 0$. We substitute these values into (3.4) and thus obtain a finite, well-defined expression for the partition function (3.4). To formulate the theorems we need the following standard result in elliptic

operator theory: There exists $\epsilon > 0$ and $\delta > 0$ such that the following heat kernel expansion holds for $0 < t < \delta$:

$$\text{Tr}(e^{-t\Delta_k}) = \sum_{0 \leq l \leq l_0} a_l(\Delta_k)t^{-l} + O(t^\epsilon) \tag{3.14}$$

Theorem 3.2. *The zeta-function $\zeta(s \,|\, |T|)$ is well-defined and analytic for $Re(s) >> 0$. Its analytic continuation to a meremorphic function on $\mathbf{C}$ is regular at $s = 0$ and satisfies the formula*

$$\zeta = \zeta(0 \,|\, |T|) = \sum_{k=0}^{N}(-1)^k(a_0(\Delta_k) - dimH^k(R(S))) \tag{3.15}$$

where $a_0(\Delta_k)$ is the coefficient corresponding to $l = 0$ in (3.14).

Proof. Let $\{\lambda_j\}$ denote the non-zero eigenvalues of T; from the definition of the zeta-function we have

$$\zeta(2s \,|\, |T|) = \sum_j \frac{1}{|\lambda_j|^{2s}} = \sum_j \frac{1}{(\lambda_j^2)^s} = \zeta(s \,|\, T^2)$$

Since $T^2 = T_0^*T_0$ it follows from lemma 3.1 that $\zeta(s \,|\, |T|) = \zeta(\frac{s}{2} \,|\, T_0^*T_0)$ is well-defined and analytic for $Re(s) >> 0$, with analytic continuation regular at $s = 0$. A simple induction argument based on (3.11) and starting with $\zeta(s \,|\, T_0^*T_0) = \zeta(s \,|\, \Delta_0) - \zeta(s \,|\, T_1^*T_1)$ gives

$$\zeta(s \,|\, T_0^*T_0) = \sum_{k=0}^{N}(-1)^k\zeta(s \,|\, \Delta_k) \,.$$

To show (3.15) it now suffices to show $\zeta(0 \,|\, \Delta_k) = a_0(\Delta_k) - dimH^k(R(S))$. We show this starting with the formula [15, p.78]

$$\zeta(s \,|\, \Delta_k) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\left(\sum_{\mu_j^{(k)} > 0} e^{-t\mu_j^{(k)}}\right) dt$$

where $\Gamma(s)$ is the gamma-function. Using $\Gamma(s) = \frac{1}{s}\Gamma(s+1)$ and $\sum_{\mu_j^{(k)} > 0} e^{-t\mu_j^{(k)}} = \text{Tr}(e^{-t\Delta_k}) - dim(ker(\Delta_k))$ we get

$$\zeta(s \,|\, \Delta_k) = \frac{s}{\Gamma(s+1)} \int_0^\infty t^{s-1}\left(\text{Tr}(e^{-t\Delta_k}) - dim(ker(\Delta_k))\right) dt \,.$$

It is well-known that $\int_\delta^\infty t^{s-1}(\mathrm{Tr}(e^{-t\Delta_k}) - dim(ker(\Delta_k)))\,dt$ is an entire function of $s \in \mathbf{C}$ for all $\delta > 0$ (this follows for example from the results of [15, §1.6]). Decomposing the integral $\int_0^\infty$ above into $\int_0^\delta + \int_\delta^\infty$ and substituting the expansion (3.14) for $\mathrm{Tr}(e^{-t\Delta_k})$ in the $\int_0^\delta$ integral leads to

$$\zeta(0\,|\,\Delta_k) = a_0(\Delta_k) - dim(ker(\Delta_k)) = a_0(\Delta_k) - dim H^k(R(S))\,.$$

This completes the proof.

Theorem 3.3. *The eta-function $\eta(s\,|\,T)$ is well-defined and analytic for $Re(s) >> 0$. Its analytic continuation to a meremorphic function on $\mathbf{C}$ is regular at $s = 0$. It satisfies $\eta(s\,|\,T) = \eta(s\,|\,D)$, where $D = T + \sum_{k=1}^N (T_k + T_k^*)$ is a selfadjoint elliptic operator on $\oplus_{k=0}^N \Gamma_k$. (Here we define $T_k = 0$ on Γ_j for $j \neq k$.)*

Proof. Since $T_k \circ T_{k+1} = 0$ we have

$$D^2 = \oplus_{k=0}^N \Delta_k\,, \tag{3.16}$$

which is elliptic, hence D is elliptic. A result due to Atiyah, Patodi and Singer [5] in the case when $dim M$ is odd, and Gilkey [15, §4.3] in the case when $dim M$ is even now states that $\eta(s\,|\,D)$ is well-defined and analytic for $Re(s) >> 0$, with analytic continuation regular at $s = 0$. We will show that $\eta(s\,|\,T) = \eta(s\,|\,D)$, which proves the theorem.

We can decompose $D = \widetilde{D} \oplus \widetilde{T}$ with $\widetilde{D} = \sum_{k=1}^N (T_k + T_k^*)$ acting on $(\oplus_{k=1}^N \Gamma_k) \oplus ker(T)$ and $\widetilde{T}$ the restriction of T to $ker(T)^\perp$, as previously. From (3.16) it follows that the zeta-function of $D^2 = \widetilde{D}^2 \oplus \widetilde{T}^2$ is well-defined for $Re(s) >> 0$, which implies that the eta-functions of D, $\widetilde{D}$, and $\widetilde{T}$ are well-defined for $Re(s) >> 0$. We will show that $\eta(s\,|\,\widetilde{D}) = 0$ for $Re(s) >> 0$, from which it follows that $\eta(s\,|\,D) = \eta(s\,|\,T)$. To show this it suffices to construct an isomorphism $\omega \mapsto \omega'$ which maps each eigenvector ω of $\widetilde{D}$ with eigenvalue λ to an eigenvector with eigenvalue $-\lambda$. We decompose $\omega \in \oplus_{k=1}^N \Gamma_k \oplus ker(T)$ as $\omega = \oplus_{k=0}^N \omega_k$, $\omega_k \in \Gamma_k$, and define the isomorphism $\omega \mapsto \omega'$ by $\omega' = \oplus_{k=0}^N (-1)^k \omega_k$. To show that this isomorphism has the required property we consider the eigenvalue equation $\widetilde{D}\omega = \lambda\omega$: It is equivalent to the collection of equations

$$\{\,T_k(\omega_k) + T_{k-1}^*(\omega_{k-2}) = \lambda\omega_{k-1}\,\} \qquad k = 1, \ldots, N+1 \tag{3.17}$$

30

(with $\omega_{-1} = 0$). From this we calculate

$$
\begin{aligned}
T_k \omega'_k + T^*_{k-1} \omega'_{k-2} &= (-1)^k \left(T_k \omega_k + T^*_{k-1} \omega_{k-2} \right) \\
&= -(-1)^{k-1} \lambda \omega_{k-1} \\
&= (-\lambda) \omega'_{k-1}.
\end{aligned}
$$

It follows from (3.17) that $\widetilde{D}\omega' = -\lambda\omega'$ as required. This completes the proof. (The statement $\eta(0\,|\,T) = \eta(0\,|\,D)$ is similar to [5, (I), proposition(4·20)] and can also be shown by a proof similar to the one given there).

3.2 Variation of the partition function under variation of inner products and resolvent

In the following the group actions on the partition function considered in §2 are generalised to the present case. Formulae for the variation of the partition function under the infinitessimal versions of these actions are derived and resulting invariance properties of the partition function are pointed out.

In the following we assume given a metric on M and Hermitian structures in the bundles ξ_k and use them to construct inner products $\langle\,\cdot\,,\,\cdot\,\rangle_k$ in the spaces Γ_k by analogy with (3.1), denoted collectively by $\langle\,\cdot\,,\,\cdot\,\rangle$. Let $GL(\xi)$ denote the group of isomorphisms of the vectorbundle ξ, i.e. $A \in GL(\xi)$ is a collection of isomorphisms $\{ A_x \in GL(\xi_x) \}_{x \in M}$ varying smoothly with x. Then $A \in GL(\xi)$ is an isomorphism of $\Gamma = \Gamma(M,\xi)$ defined by $A(\omega)(x) = A_x(\omega(x))$. The group $GL(\xi_N) \times \cdots \times GL(\xi_1) \times GL(\xi)$ acts on the inner products $\langle\,\cdot\,,\,\cdot\,\rangle$ in the spaces $\Gamma_k = \Gamma(M,\xi_k)$ by analogy with the action described in §2 (see (2.2)).

Lemma 3.4. *All inner products in the spaces Γ_k constructed as in (3.1) from a metric on M and Hermitian structures in the bundles ξ_k can be obtained from the group action above on the inner products constructed from the given metric and Hermitian structures.*

Proof. Let *vol* and *vol'* be the volume forms on M constructed from 2 different metrics, and let $\{\langle\,\cdot\,,\,\cdot\,\rangle_x\}_{x \in M}$ and $\{\langle\,\cdot\,,\,\cdot\,\rangle_x'\}_{x \in M}$ be 2 different Hermitian structures in ξ_k. Then $vol'(x) = f(x)vol(x)$ and $\langle u, v\rangle_x' = \langle u, Q_x v\rangle_x$ where f is a smooth strictly positive function on M and Q_x is a map on $(\xi_k)_x$ strictly positive and Hermitian w.r.t. $\langle\,\cdot\,,\,\cdot\,\rangle_x$, varying smoothly with $x \in M$. Then the inner products $\langle\,\cdot\,,\,\cdot\,\rangle$ and $\langle\,\cdot\,,\,\cdot\,\rangle'$ in Γ_k constructed by analogy with (3.1) are related by $\langle \omega, \tau \rangle' = \langle \omega, (fQ)\tau \rangle$ where $(Q\tau)(x) = Q_x(\tau(x))$. It follows that

$\langle \cdot , \cdot \rangle'$ is obtained from the action of $A = (fQ)^{1/2}$ on $\langle \cdot , \cdot \rangle$ (with $Q_x^{1/2}$ defined via spectral theory).

We will derive the variation of the modulus of the partition function $Z(\beta ; R(S), \langle \cdot , \cdot \rangle_H, \langle \cdot , \cdot \rangle)$, given by (3.4) under the infinitessimal version of the above group action on the inner products $\langle \cdot , \cdot \rangle$. We begin by deriving the variation of $\widetilde{Z}(R(S), \langle \cdot , \cdot \rangle_H, \langle \cdot , \cdot \rangle)$, given by (3.5), under the infinitessimal version of the group action. To do this we need the variation of $\widetilde{Z}(R(S), \langle \cdot , \cdot \rangle)$, given by (3.6); this is readily obtained from a theorem in [25]. Let $A(u) = (A_N(u), \ldots, A_1(u), A_0(u))$ be a smooth curve in $GL(\xi_N) \times \cdots \times GL(\xi_1) \times GL(\xi)$ with $A(0) = I$ (the identity). We define the endomorphisms $B_k \in End(\xi_k)$ for $k = 0, 1, \ldots, N$ by $B_k = \frac{d}{du}\big|_{u=0}(A_k^*(u)A_k(u))$. These induce linear maps $B_k : \Gamma_k \longrightarrow \Gamma_k$ with the property

$$\frac{d}{du}\bigg|_{u=0} \langle \omega, \tau \rangle_k^{(A(u))} = \langle \omega, B_k \tau \rangle_k$$

Let $B \in End(\xi_k)$ be arbitrary. From elliptic operator theory we have the following asymptotic expansion for $t \to 0_+$:

$$\text{Tr}(B\, e^{-t\Delta_k}) = \sum_{0 \le l \le l_0} a_l(B \,|\, \Delta_k)\, t^{-l} + O(t^\epsilon) \qquad (3.18)$$

for some $\epsilon > 0$. By Hodge-theory the spaces

$$\mathcal{H}_k = ker(T_k) \cap Im(T_{k+1})^\perp = ker(\Delta_k)$$

are finite-dimensional. Let $P_{[\mathcal{H}_k]}$ and $P_{[\mathcal{H}_k^\perp]}$ denote the orthogonal projections of the completion of Γ_k w.r.t. $\langle \cdot , \cdot \rangle_k$ onto $\mathcal{H}_k$ and $\mathcal{H}_k^\perp$ respectively. Then we have

$$
\begin{aligned}
\text{Tr}(B\, e^{-t\Delta_k}) &= \text{Tr}(B\, P_{[\mathcal{H}_k^\perp]}\, e^{-t\Delta_k}) + \text{Tr}(B\, P_{[\mathcal{H}_k]}\, e^{-t\Delta_k}) \\
&= \text{Tr}(B\, P_{[\mathcal{H}_k^\perp]}\, e^{-t\Delta_k}) + \text{Tr}(B\, P_{[\mathcal{H}_k]}).
\end{aligned}
$$

Combining this with (3.18) we get the following asymptotic expansion for $t \to 0_+$:

$$\text{Tr}(B\, P_{[\mathcal{H}_k^\perp]}\, e^{-t\Delta_k}) = \sum_{0 \le l \le l_0} a_l(B\, P_{[\mathcal{H}_k^\perp]} \,|\, \Delta_k)\, t^{-l} + O(t^\epsilon) \qquad (3.19)$$

where

$$a_0(B\,P_{[\mathcal{H}_k^\perp]}\,|\,\Delta_k) \;=\; a_0(B\,|\,\Delta_k) - \mathrm{Tr}(B\,P_{[\mathcal{H}_k]}) \qquad (3.20)$$
$$a_l(B\,P_{[\mathcal{H}_k^\perp]}\,|\,\Delta_k) \;=\; a_l(B\,|\,\Delta_k) \qquad \text{for } l > 0\,.$$

The following theorem is a simple reformulation of theorem $1'$ in [25]. It expresses the variation of $\widetilde{Z}(R(S), \langle\,\cdot\,,\cdot\,\rangle)$ under the infinitessimal action of $GL(\xi_N) \times \cdots \times GL(\xi_1) \times GL(\xi)$ on the inner products in the spaces Γ_k. ($A(u)$ and B_k are as above).

Theorem 3.5. (Due to A. Schwartz [25]). *The expression $\widetilde{Z}(R(S), \langle\,\cdot\,,\cdot\,\rangle)$ given by the zeta-regularisation of (3.6) satisfies*

$$\frac{d}{du}\Big|_{u=0} \widetilde{Z}(R(S), \langle\,\cdot\,,\cdot\,\rangle^{A(u)}) = \left(\sum_{k=0}^{N} \frac{1}{2}(-1)^k a_0(B_k\,P_{[\mathcal{H}_k^\perp]}\,|\,\Delta_k) \right) \widetilde{Z}(R(S), \langle\,\cdot\,,\cdot\,\rangle)$$
$$(3.21)$$

If the cohomology of the resolvent vanishes then from (3.4) with $\beta = 1$ we see that $\widetilde{Z}(R(S), \langle\,\cdot\,,\cdot\,\rangle) = |Z(R(S), \langle\,\cdot\,,\cdot\,\rangle)|$ is the modulus of the partition function of S. In this case we obtain from theorem 3.5 the following expression for the variation of the modulus of the partition function of S under the infinitessimal group action on the inner products.

Corollary 3.6. *Assume the cohomology of the resolvent vanishes. Then the modulus $|Z(R(S), \langle\,\cdot\,,\cdot\,\rangle)|$ of the partition function of S with $\beta = 1$ satisfies*

$$\frac{d}{du}\Big|_{u=0} |Z(R(S), \langle\,\cdot\,,\cdot\,\rangle^{A(u)})| = \left(\sum_{k=0}^{N} \frac{1}{2}(-1)^k a_0(B_k\,|\,\Delta_k) \right) |Z(R(S), \langle\,\cdot\,,\cdot\,\rangle)|$$
$$(3.22)$$

The formula (3.22) is a generalisation of the formula (2.18) for the finite-dimensional case. Indeed, if the operators in the present case were acting on finite-dimensional vectorspaces we would have

$$\text{`` } a_0(B_k\,|\,\Delta_k) = \lim_{t\to 0} \mathrm{Tr}(B_k e^{-t\Delta_k}) = \mathrm{Tr}(B_k)\,. \text{ ''} \qquad (3.23)$$

Substituting this into (3.22) would give (2.18).

When the dimension of M is odd the modulus of the partition function has invariance properties which are considerably more general that in the

33

finite-dimensional case. This is due to the following remarkable (although well-known) result.

Theorem 3.7. *If the dimension of M is odd then $a_0(B \mid \Delta_k) = 0$ in the asymptotic expansion (3.18) for all $B \in End(\xi_k)$ for all $k = 0, 1, \ldots, N$.*

This is a standard result in elliptic operator theory (see e.g. [15, §1.7 lemma 1.7.4(d)]). Combining this theorem with corollary 3.6 and lemma 3.4 leads to the following theorem. (The statements in the theorem concerning independence of metric and Hermitian structure were shown by Schwarz in [25]). We say that S and $R(S)$ are *topological* if their definitions do not depend on choices of Hermitian structures in the bundles ξ_k and metric on M.

Theorem 3.8. *Assume the cohomology of the resolvent vanishes. If $\dim M$ is odd then the modulus $|Z(R(S), \langle \cdot, \cdot \rangle)|$ of the partition function of S with $\beta = 1$ is invariant under the action of $GL(\xi_N) \times \cdots \times GL(\xi_1) \times GL(\xi)$ on the inner products in the spaces Γ_k. If S and $R(S)$ are topological then $|Z(R(S), \langle \cdot, \cdot \rangle)|$ is independent of the choice of the metric on M and Hermitian structures in the bundles ξ_k.*

Proof. We must show that $|Z(R(S), \langle \cdot, \cdot \rangle^A)| = |Z(R(S), \langle \cdot, \cdot \rangle)|$ for arbitrary $A = (A_N, \ldots, A_1, A_0) \in GL(\xi_N) \times \cdots \times GL(\xi_1) \times GL(\xi)$. For $s \in [0, 1]$ set $Q_k(s) = ((1-s) \cdot I_k + s \cdot A_k^* A_k)^{1/2} \in GL(\xi_k)$. ($(Q_k(s))_x$ is the square root of a map on $(\xi_k)_x$ which is strictly positive and selfadjoint w.r.t. $(\langle \cdot, \cdot \rangle_k)_x$, and is therefore well-defined via spectral theory). Then $Q(s) = (Q_N(s), \ldots, Q_1(s), Q_0(s))$ is a smooth curve in $GL(\xi_N) \times \cdots \times GL(\xi)$ with $\langle \cdot, \cdot \rangle^{Q(0)} = \langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle^{Q(1)} = \langle \cdot, \cdot \rangle^A$. It now suffices to show that $\frac{d}{ds}|Z(R(S), \langle \cdot, \cdot \rangle^{Q(s)})| = 0$. Set $Q(s, u) = Q(s)^{-1} Q(s + u)$, then $\langle \cdot, \cdot \rangle^{Q(s+u)} = \langle \cdot, \cdot \rangle^{Q(s)Q(s,u)} = (\langle \cdot, \cdot \rangle^{Q(s)})^{Q(s,u)}$. It follows from (3.22) and theorem 3.7 that

$$\frac{d}{ds}|Z(R(S), \langle \cdot, \cdot \rangle^{Q(s)})| = \frac{d}{du}\Big|_{u=0}|Z(R(S), (\langle \cdot, \cdot \rangle^{Q(s)})^{Q(s,u)})| = 0.$$

This completes the proof.

We will now show that the above symmetry properties of the modulus of the partition function continue to hold in the general case where the cohomology of the resolvent is not required to vanish.

Theorem 3.9

(i) The modulus of the partition function (3.4) with $\beta = 1$ satisfies

$$\frac{d}{du}\Big|_{u=0}|Z(R(S),\langle\cdot\,,\cdot\rangle_H,\langle\cdot\,,\cdot\rangle^{A(u)})|$$

$$= \Big(\sum_{k=0}^{N}\frac{1}{2}(-1)^k a_0(B_k\,|\,\Delta_k)\Big)\,|Z(R(S),\langle\cdot\,,\cdot\rangle_H,\langle\cdot\,,\cdot\rangle)|\,. \qquad (3.24)$$

(ii) If $dim M$ is odd the modulus of the general partition function (3.4) satisfies

$$|Z(\beta\,;R(S),\langle\cdot\,,\cdot\rangle_H,\langle\cdot\,,\cdot\rangle)| = |\beta|^{-\zeta/2}|Z(R(S),\langle\cdot\,,\cdot\rangle_H,\langle\cdot\,,\cdot\rangle)| \qquad (3.25)$$

with

$$\zeta = -\sum_{k=0}^{N}(-1)^k dim H^k(R(S)) \qquad (3.26)$$

and is invariant under the action of $GL(\xi_N) \times \cdots \times GL(\xi_1) \times GL(\xi)$ on $\langle\cdot\,,\cdot\rangle$. It is independent of the choice of metric on M and Hermitian structures in the bundles ξ_k when S and $R(S)$ are topological.

Proof. The formulae (3.25) and (3.26) in (ii) follow from the expression (3.4) and theorems 3.2 and 3.7. The remainder of (ii) follows from (i) in the same way that theorem 3.8 was obtained from corollary 3.6. The proof of (i) is based on the following formula, where Ψ is given by (3.7):

$$\frac{d}{du}\Big|_{u=0}\Psi(R(S),\langle\cdot\,,\cdot\rangle_H,\langle\cdot\,,\cdot\rangle^{A(u)})$$

$$= \Big(\sum_{k=0}^{N}\frac{1}{2}(-1)^k \mathrm{Tr}(P_{[\mathcal{H}_k]}B_k)\Big)\,\Psi(R(S),\langle\cdot\,,\cdot\rangle_H,\langle\cdot\,,\cdot\rangle)\,. \qquad (3.27)$$

Combining this with theorem 3.5 and (3.20) gives (3.24), proving the theorem. The following derivation of (3.27) is inspired by, and modeled on the derivation of [21, formula (3.4)]. We set

$$X(u) = (\Phi_{k(u)}^{-1})^{*(u)}\Phi_{k(u)}^{-1} : H^k(R(S)) \xrightarrow{\simeq} H^k(R(S))\,,$$

where $\Phi_{k(u)} : \mathcal{H}_{k(u)} \xrightarrow{\simeq} H^k(R(S))$ is defined as in (2.23) and (2.24) with the inner product $\langle\cdot\,,\cdot\rangle_k$ in Γ_k replaced by $\langle\cdot\,,\cdot\rangle_k^{A_k(u)}$, and $(\cdot)^{*(u)}$ denotes the

adjoint map determined by $\langle \cdot, \cdot \rangle_k^{A_k(u)}$ and $\langle \cdot, \cdot \rangle_{H^k}$. To show (3.27) it suffices to show that

$$\frac{d}{du}\Big|_{u=0} det X(u) = \text{Tr}(P_{[\mathcal{H}_k]} B_k) \, det(X(0)). \tag{3.28}$$

We see by differentiating the formula $log \, det(X) = \text{Tr}(log(X))$ at $u = 0$ that (3.28) is equivalent to

$$\text{Tr}\Big(X(0)^{-1} X'(0)\Big) = \text{Tr}\Big(P_{[\mathcal{H}_k]} B_k\Big). \tag{3.29}$$

Pick a basis (h_j) for $H^k(R(S))$, orthonormal w.r.t. $\langle \cdot, \cdot \rangle_{H^k}$, and set $f_j(u) = \Phi_{k(u)}^{-1}(h_j)$. Let (η_j) be a basis for $\mathcal{H}_k$, orthonormal w.r.t. $\langle \cdot, \cdot \rangle_k$, and define the matrix Y by $f_j(0) = \sum_r Y_{jr} \eta_r$. Then

$$\begin{aligned} X_{jl} &\equiv \langle h_j, X(0) h_l \rangle_{H^k} = \langle f_j(0), f_l(0) \rangle_k \\ &= \sum_{r,s} \langle Y_{jr} \eta_r, Y_{ls} \eta_s \rangle_k = \sum_r Y_{jr} Y_{lr} = (YY^*)_{jl}. \end{aligned} \tag{3.30}$$

Defining the matrices X_{jl}^{-1} and $X'(0)_{jl}$ by $\langle h_j, X(0)^{-1} h_l \rangle_{H^k}$ and $\langle h_j, X'(0) h_l \rangle_{H^k}$ respectively it follows from (3.30) that

$$\begin{aligned} \text{Tr}\Big(X(0)^{-1} X'(0)\Big) &= \sum_{j,l} X_{jl}^{-1} X'(0)_{jl} = \sum_{j,l} (YY^*)_{jl}^{-1} X'(0)_{jl} \\ &= \sum_{j,l,r} Y_{rj}^{-1} Y_{rl}^{-1} X'(0)_{jl}. \end{aligned}$$

Also, we have

$$\text{Tr}\Big(P_{[\mathcal{H}_k]} B_k\Big) = \sum_r \langle \eta_r, B_k \eta_r \rangle_k = \sum_{r,j,k} Y_{rj}^{-1} Y_{rl}^{-1} \langle f_j(0), B_k f_l(0) \rangle_k.$$

Therefore, to show (3.29) it suffices to show that

$$X'(0)_{jl} = \langle f_j(0), B_k f_l(0) \rangle_k \tag{3.31}$$

We calculate

$$\begin{aligned} X'(0)_{jl} &= \frac{d}{du}\Big|_{u=0} \langle h_j, X(u) h_l \rangle_{H^k} = \frac{d}{du}\Big|_{u=0} \langle f_j(u), f_l(u) \rangle_k^{A_k(u)} \\ &= \lim_{u \to 0} \frac{1}{u} \Big(\langle f_j(u), f_l(u) \rangle_k^{A_k(u)} - \langle f_j(0), f_l(0) \rangle_k \Big). \end{aligned} \tag{3.32}$$

36

Now, since

$$ker(T_k) = Im(T_{k+1}) \oplus \mathcal{H}_k = Im(T_{k+1}) \oplus_{A_k(u)} \mathcal{H}_{k(u)} \,,$$

where $\oplus_{A_k(u)}$ denotes direct sum orthogonal w.r.t. $\langle \cdot \,, \cdot \rangle_k^{A_k(u)}$, we can uniquely write

$$f_j(u) = f_j(0) + w_j(u) \;,\;\; w_j(u) \in Im(T_{k+1})$$

and get

$$\langle f_j(u) \,, f_l(u) \rangle_k^{A_k(u)} = \langle f_j(0) \,, f_l(u) \rangle_k^{A_k(u)} = \langle f_j(0) \,, A_k^*(u) A_k(u) f_l(u) \rangle_k$$

and

$$\langle f_j(0) \,, f_l(0) \rangle_k = \langle f_j(0) \,, f_l(u) \rangle_k \,.$$

Substituting this into (3.32) gives

$$X'(0)_{jl} = \lim_{u \to 0} \frac{1}{u} \Big(\langle f_j(0) \,, (A_k^*(u) A_k(u) - I) f_l(u) \rangle_k \Big) = \langle f_j(0) \,, B_k f_l(0) \rangle$$

proving (3.31) and thereby (3.27). This completes the proof.

Note from (3.26) that when $dim M$ is odd the quantity $\zeta = \zeta(0 \,|\, T)$ in the expression (3.4) for the partition function is independent of metric on M and Hermitian structure in ξ. However, the quantity $\eta = \eta(0 \,|\, T)$ in (3.4), and therefore the phase of the partition function, does depend on the metric and Hermitian structure in general. (This was shown explicitly by Witten in [27, §2] for partition functions in the semiclassical approximation for the Witten-invariant). For the class of quadratic functionals considered in §4 we will apply the Atiyah-Patodi-Singer index theorem to derive a formula involving η.

From (3.23) we see that (3.24) is a generalisation to the present infinite-dimensional case of the formula (2.32) for the finite-dimensional case.

We now extend the group action on the pair $(R(S), \langle \cdot \,, \cdot \rangle_{H(R(S))})$ considered in §2 to the present case and derive the variation of the partition function under the infinitessimal version of this action. In the present case we take the group to be $GL(\xi_N) \times \cdots \times GL(\xi_1)$. Each $C_k \in GL(\xi_k)$ induces a linear map on Γ_k which we also denote by C_k. The right group action of $GL(\xi_N) \times \cdots \times GL(\xi_1)$ on the set of resolvents $R(S)$ and collection $\langle \cdot \,, \cdot \rangle_H$ of inner products in the cohomology spaces $H(R(S))$ is defined in the same way

as in §2. It is straightforward to see that the subset consisting of elliptic resolvents is invariant under this action. (This follows from lemma 10 in [25]). In the following we restrict our considerations to this subset. The group $GL(\xi_N) \times \cdots \times GL(\xi_1)$ also acts on the inner products $\langle \cdot, \cdot \rangle$ in the spaces Γ_k as before. (The action on $\langle \cdot, \cdot \rangle_0$ is trivial).

A version of theorem 2.6 continues to hold in the present infinite-dimensional case. Indeed, the calculations leading to (2.35) and (2.38) continue to hold and it follows that $(T_k^C)^{*(C)}T_k^C = C_k^{-1}T_k^*T_kC_k$ and $(\Phi_k^C)^{*(C)}\Phi_k^C = C_k^{-1}\Phi_k^*\Phi_kC_k$ have the same eigenvalues (with same multiplicities) as $T_k^*T_k$ and $\Phi_k^*\Phi_k$ respectively. Therefore the zeta-functions of $(T_k^C)^{*(C)}T_k^C$ and $(\Phi_k^C)^{*(C)}\Phi_k^C$ are equal to those of $T_k^*T_k$ and $\Phi_k^*\Phi_k$ respectively, so their zeta-regularised determinants are the same. Since $\langle \cdot, \cdot \rangle_0$ is unchanged by the action of $GL(\xi_N) \times \cdots \times GL(\xi_1)$ on $\langle \cdot, \cdot \rangle$ the map $\tilde{T}$ is unchanged, so from (3.4)–(3.7) we get

Theorem 3.10. *For all $C = (C_N, \ldots, C_1) \in GL(\xi_N) \times \cdots \times GL(\xi_1)$ the partition function (3.4) satisfies*

$$Z(\beta; R(S)^C, \langle \cdot, \cdot \rangle_H^C, \langle \cdot, \cdot \rangle^C) = Z(\beta; R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) \tag{3.33}$$

The variation of $Z(\beta; R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle)$ under the variation of the pair $(R(S), \langle \cdot, \cdot \rangle_H)$ by the above group action can now be obtained by combining theorem 3.10 with theorem 3.9 and noting that ζ and η in (3.4) are independent of the choice of resolvent $R(S)$. Let $C(u)$ be a smooth curve in $GL(\xi_N) \times \cdots \times GL(\xi_1)$ with $C(0) = I$ and set $D_k = \frac{d}{du}\big|_{u=0}(C_k^*(u)C_k(u))$. The induced linear maps on Γ_k are also denoted by D_k.

Corollary 3.11. *The partition function (3.4) satisfies*

$$\frac{d}{du}\bigg|_{u=0} Z(\beta; R(S)^{C(u)}, \langle \cdot, \cdot \rangle_H^{C(u)}, \langle \cdot, \cdot \rangle)$$
$$= \bigg(\sum_{k=1}^N \frac{1}{2}(-1)^{k+1} a_0(D_k \,|\, \Delta_k)\bigg) Z(\beta; R(S), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle). \tag{3.34}$$

If $\dim M$ is odd the partition function is invariant under the group action of $GL(\xi_N) \times \cdots \times GL(\xi_1)$ on the pair $(R(S), \langle \cdot, \cdot \rangle_H)$.

Theorem 3.9(i) and corollary 3.11 above are extensions of results of Schwarz, theorems 1′ and 2′ in [25], derived for the restricted case where

the cohomology of the resolvent vanishes. The derivations of both of these results in [25] use elliptic operator theory (heat kernel expansion), however the preceding shows that they are related by the straightforward algebraic result theorem 3.10.

Note that the formula (3.34) is a generalisation to the present infinite-dimensional case of the formula (2.41) for the finite-dimensional case.

4 Examples and Applications

4.1 A general example of a quadratic functional with elliptic resolvent

We now present a class of topological quadratic functionals of the type discussed in §3. Each of these functionals has a canonical topological elliptic resolvent. The corresponding partition functions include the partition functions appearing in the semiclassical approximation (1.14) for the Witten invariant. We will apply the method and results of §3 to evaluate these partition functions. As a byproduct we will show that the metric-dependence of the Ray-Singer torsion factors out in a simple way as a power of the volume of M in certain cases where the cohomology is non-vanishing.

Throughout the following we assume that the dimension n of the manifold M (introduced in §3) is odd. Let $\chi \colon \pi_1(M) \to O(V, \langle \cdot, \cdot \rangle_V)$ be a representation of $\pi_1(M)$ on a real vectorspace V orthogonal w.r.t. an inner product $\langle \cdot, \cdot \rangle_V$ in V. The by a standard construction in differential geometry χ determines a real flat vectorbundle ξ over M and a flat connection map ∇ on the space $\Omega(M, \xi)$ of differential forms on M with values in ξ (i.e. the space of smooth sections in the vectorbundle $\Lambda(TM)^* \otimes \xi$). Let ∇_q denote the restriction of ∇ to the space $\Omega^q(M, \xi)$ of q-forms and define the cohomology spaces

$$H^q(\nabla) = ker(\nabla_q) \big/ Im(\nabla_{q-1}) \,.$$

The bundle ξ has a canonical Hermitian structure (determined by $\langle \cdot, \cdot \rangle_V$) which we denote by $\{\langle \cdot, \cdot \rangle_x\}_{x \in M}$ which ∇ is compatible with. This Hermitian structure determines for each $x \in M$ a linear map $\langle \cdot \rangle_x \colon \xi_x \otimes \xi_x \longrightarrow R$, which in turn determines linear maps

$$\left(\Lambda^p(T_xM)^* \otimes \xi_x \right) \otimes \left(\Lambda^q(T_xM)^* \otimes \xi_x \right) \overset{\wedge}{\longrightarrow} \Lambda^{p+q}(T_xM)^* \otimes \left(\xi_x \otimes \xi_x \right) \overset{\langle \cdot \rangle_x}{\longrightarrow}$$
$$\longrightarrow \Lambda^{p+q}(T_xM)^* \,.$$

We denote the image of $\omega_x \otimes \tau_x$ under this map by $\langle \omega_x \wedge \tau_x \rangle_x$.

Set $n = dimM$ and define m by $n = 2m + 1$. Assume to begin with that m is odd; in this case we define the quadratic functional S_∇ on $\Omega^m(M, \xi)$ by

$$S_\nabla(\omega) = \int_M \langle \omega(x) \wedge (\nabla_m \omega)(x) \rangle_x \qquad (4.1)$$

where ∇_m is the restriction of ∇ to $\Omega^m(M, \xi)$. Choosing a metric on M we can construct from the metric and Hermitian structure in ξ a Hermitian structure in $\Lambda(T_x M)^* \otimes \xi$ and from this we get inner products $\langle \cdot, \cdot \rangle_q$ in the spaces $\Omega^q(M, \xi)$ by analogy with (3.1). This enables us to write

$$S_\nabla(\omega) = \langle \omega, T_\nabla \omega \rangle_m \quad , \quad T_\nabla = *\nabla_m \qquad (4.2)$$

where $*$ is the Hodge-star map. The map T_∇ is formally selfadjoint with the property

$$T_\nabla^2 = \nabla_m^* \nabla_m . \qquad (4.3)$$

There is a canonical topological elliptic resolvent $R(S_\nabla)$ for the functional (4.1):

$$0 \overset{0}{\longrightarrow} \Omega^0(M, \xi) \overset{\nabla_0}{\longrightarrow} \ldots \overset{\nabla_{m-2}}{\longrightarrow} \Omega^{m-1}(M, \xi) \overset{\nabla_{m-1}}{\longrightarrow} ker(S_\nabla) \overset{0}{\longrightarrow} 0 \qquad (4.4)$$

Comparing (4.4) with (2.5) we see that for the resolvent $R(S_\nabla)$ we have $N = m$, $\Gamma_k = \Omega^{m-k}(M, \xi)$, $T_k = \nabla_{m-k}$ and $H^k(R(S_\nabla)) = H^{m-k}(\nabla)$. Note that by our previously established notation convention $ker(S_\nabla) \equiv ker(T_\nabla) = ker(\nabla_m)$.

Now choose an inner product $\langle \cdot, \cdot \rangle_{H^k}$ in each space $H^k(R(S_\nabla))$, then from (3.4) we find that the partition function of S_∇ with the resolvent (4.4) is :

$$Z(\beta \, ; R(S_\nabla), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) = \pi^{\zeta/2} \, e^{-\frac{i\pi}{4}((\frac{2\theta}{\pi} \mp 1)\zeta \pm \eta)} \, |\beta|^{-\zeta/2} \, \tau(M, \chi, \langle \cdot, \cdot \rangle_H)^{1/2} \qquad (4.5)$$

where we define for m both odd and even :

$$\tau(M, \chi, \langle \cdot, \cdot \rangle_H) = \widetilde{\tau}(M, \chi, g) \cdot \Psi(M, \chi, g, \langle \cdot, \cdot \rangle_H)^2 \qquad (4.6)$$

with

$$\widetilde{\tau}(M, \chi, g) = \prod_{q=0}^{m-1} det(\widetilde{\nabla}_q^* \widetilde{\nabla}_q)^{(-1)^q} det(\widetilde{\nabla}_m^* \widetilde{\nabla}_m)^{\frac{1}{2}(-1)^m} \qquad (4.7)$$

40

and

$$\Psi(M, \chi, g, \langle \cdot, \cdot \rangle_H) = \prod_{q=0}^{m} det(\phi_q^* \phi_q)^{\frac{1}{2}(-1)^q} \tag{4.8}$$

where $\phi_q = \Phi_{m-q} : ker(\Delta_q) \xrightarrow{\simeq} H^q(\nabla)$ is the restriction of the projection map $ker(\nabla_q) \to H^q(\nabla)$ to $ker(\Delta_q)$. In particular, for $\beta = i\lambda$, $\lambda \in \mathbf{R}_{\pm}$, we have $\theta = \pm \pi/2$ and

$$Z(i\lambda; R(S_\nabla), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) = \pi^{\zeta/2} e^{\mp \frac{i\pi}{4}\eta} k^{-\zeta/2} \tau(M, \chi, \langle \cdot, \cdot \rangle_H)^{1/2}. \tag{4.9}$$

We will discuss the quantities $\tau(M, \chi, \langle \cdot, \cdot \rangle_H)$, ζ, and η appearing in the above expressions below. First we give a treatment similar to the preceding for the case where m is even. In this case the functional (4.1) is identically zero; to see this note that

$$\langle \omega(x) \wedge (\nabla_m \omega)(x) \rangle_x = \langle (\nabla_m \omega)(x) \wedge \omega(x) \rangle_x$$

which gives

$$\begin{aligned} d\langle \omega(x) \wedge \omega(x) \rangle_x &= \langle (\nabla_m \omega)(x) \wedge \omega(x) \rangle_x + (-1)^m \langle \omega(x) \wedge (\nabla_m \omega)(x) \rangle_x \\ &= (1 + (-1)^m) \langle \omega(x) \wedge (\nabla_m \omega)(x) \rangle_x \end{aligned}$$

and the integral of this over M vanishes by Stokes' theorem. However, when m is even it is possible to define a non-vanishing functional similar to (4.1), the partition function of which is given by an expression similar to (4.5): In the preceding we replace ξ by the complex vectorbundle $\xi \otimes \mathbf{C}$ and define the real-valued quadratic functional $S_\nabla^{\mathbf{C}}(\omega)$ on $\Omega^m(M, \xi \otimes \mathbf{C})$ by

$$S_\nabla^{\mathbf{C}} = i \int_M \langle \overline{\omega(x)} \wedge (\nabla_m \omega)(x) \rangle_x.$$

This can be written as

$$S_\nabla^{\mathbf{C}}(\omega) = \langle \omega, T_\nabla \omega \rangle_m \quad , \quad T_\nabla = i * \nabla_m \tag{4.10}$$

where T_∇ is formally selfadjoint and has the same property (4.3) as in the case where m is odd: $T_\nabla^2 = \nabla_m^* \nabla_m$. Replacing ξ by $\xi \otimes \mathbf{C}$ in (4.4) we obtain an elliptic resolvent $R(S_\nabla^{\mathbf{C}})$ for $S_\nabla^{\mathbf{C}}$, and from it we obtain the following

41

expression for the partition function of $S_\nabla^{\mathbf{C}}$ (c.f. the remarks at the end of §2) :

$$Z(\beta \, ; R(S_\nabla^{\mathbf{C}}), \langle \cdot , \cdot \rangle_H, \langle \cdot , \cdot \rangle) = \pi^\zeta \, e^{-\frac{i\pi}{2}((\frac{2\theta}{\pi} \mp 1)\zeta \pm \eta)} \, |\beta|^{-\zeta} \tau(M, \chi, \langle \cdot , \cdot \rangle_H)^{-1}$$
(4.11)

with $\tau(M, \chi, \langle \cdot , \cdot \rangle_H)$ defined by (4.6).

In [25] the quadratic functional on $\Omega(M, \xi) = \oplus_{q=0}^n \Omega^q(M, \xi)$ defined by replacing ∇_m by $\nabla = \oplus_{q=0}^n \nabla_q$ in (4.1) was considered. It was shown to have a canonical topological elliptic resolvent and the modulus of its partition function was shown in the case where the cohomology vanishes to be proportional to a power of the Ray-Singer torsion of ∇. The resolvent is somewhat more involved than the resolvent (4.4) and we will not describe it here. The partition function of this functional can be evaluated by our methods; the resulting expression is equal to (4.5) for m odd and the square root of (4.11) for m even.

Examples of other quadratic functionals with elliptic resolvents were given in [25]. Our methods and results also apply for these. However, they are of a less general nature than the functional (4.1) above (since their construction requires choosing differential forms on the manifold) and we will not discuss them here.

The quantity ζ appearing in the partition functions above can be expressed in terms of the dimensions of the cohomology spaces of ∇: Since $H^k(R(S_\nabla)) = H^{m-k}(\nabla)$ for the resolvent (4.4) it follows from (3.26) with $N = m$ that

$$\zeta = (-1)^{m+1} \sum_{q=0}^m (-1)^q dim H^q(\nabla) \, .$$
(4.12)

We now discuss the quantities (i) $\tau(M, \chi, \langle \cdot , \cdot \rangle_H)$ and (ii) η appearing in the partition functions above.

(i) It follows from theorem 3.9 that $\tau(M, \chi, \langle \cdot , \cdot \rangle_H)$ is independent of the choice of metric g on M. In fact the quantity $\widetilde{\tau}(M, \chi, g)$ defined in (4.7) is precisely the Ray-Singer torsion [20] of the representation χ of $\pi_1(M)$ constructed using the metric g, and $\tau(M, \chi, \langle \cdot , \cdot \rangle_H)$ is a version of the Ray-Singer torsion as a "function of the cohomology" defined and shown to be metric-independent in [21, §3]. By an argument analogous to that following (2.26) we see that given a choice of orientation in each space $H^q(\nabla)$, $q = 0, 1, \ldots, m$ then considered as a functional of $\langle \cdot , \cdot \rangle_H$ we can interpret $\tau(M, \chi, \langle \cdot , \cdot \rangle_H)^{1/2}$

as an element
$$\tau^{1/2}(M,\chi) \in \otimes_{q=0}^{m} \Lambda^{max} H^q(\nabla)^{*q} . \qquad (4.13)$$

We will now show that there is a canonical choice of (metric-independent) inner product $\langle \cdot, \cdot \rangle_{H^0(\nabla)}$ in $H^0(\nabla)$ and use this to calculate $\tau(M,\chi,\langle \cdot, \cdot \rangle_H)$ in the case where $H^0(\nabla) \neq 0$ and $H^q(\nabla) = 0$ for $q = 1, \ldots, m$. The result shows that in this case the metric-dependence of the Ray-Singer torsion $\tilde{\tau}(M,\chi,g)$ factors out in a simple way as a power of the volume of M. The space $H^0(\nabla)$ is $ker(\nabla_0)$, i.e. the kernel of ∇ restricted to the space of sections in ξ, and can be described as follows. Choose a point $x_0 \in M$ then $H^0(\nabla)$ can be identified with the subspace $W \subseteq \xi_{x_0}$ having the property that the parallel transport of any $w \in W$ to any fibre ξ_x is independent of the choice of curve from x_0 to x. (For simplicity we have assumed that M is connected; the generalisation to the case where M has more than one connected component is obvious). The inner product $\langle \cdot, \cdot \rangle_{x_0}$ in ξ_{x_0} induces an inner product $\langle \cdot, \cdot \rangle_W$ in W, thereby determining an inner product in $H^0(\nabla) \simeq W$. Since $ker(\nabla_0) = H^0(\nabla)$ the map $\phi_0 \colon ker(\nabla_0) \xrightarrow{\simeq} H^0(\nabla)$ in (4.8) is the identity map. However, since $ker(\nabla_0)$ and $H^0(\nabla)$ are equipped with different inner products the map ϕ_0^* is not the identity. To determine ϕ_0^* we identify $H^0(\nabla)$ with W so that ϕ_0 is the map $\phi_0 \colon ker(\nabla_0) \xrightarrow{\simeq} W$ defined by

$$\phi_0^{-1}(w)(x) = \text{parallel transport of } w \text{ from } \xi_{x_0} \text{ to } \xi_x \text{ along any curve.}$$

Since ∇ is compatible with the Hermitian structure in ξ we have

$$\langle \phi_0^{-1}(w)(x), \phi_0^{-1}(v)(x) \rangle_x = \langle w, v \rangle_{x_0} = \langle w, v \rangle_W$$

for all w, $v \in W$ and $x \in M$, and it follows that

$$
\begin{aligned}
\langle \phi_0^{-1}(w), \phi_0^{-1}(v) \rangle_{\Omega^0(M,\xi)} &= \int_M \langle \phi_0^{-1}(w)(x), \phi_0^{-1}(v)(x) \rangle_x vol(x) \\
&= \langle w, v \rangle_W V(M)
\end{aligned}
$$

where $V(M)$ is the volume of M determined by the metric g. It follows that $(\phi_0^{-1})^* \phi_0^{-1} = V(M) \cdot I$, and therefore $det(\phi_0^* \phi_0)^{-1} = V(M)^{dim H^0(\nabla)}$. It now follows from (4.6) and (4.8) that for the case we are considering,

$$\tau(M,\chi,\langle \cdot, \cdot \rangle_H) = \tilde{\tau}(M,\chi,g) \cdot V(M)^{-dim H^0(\nabla)} . \qquad (4.14)$$

This shows the following

43

Theorem 4.1. *Assume* $H^0(\nabla) \neq 0$ *and* $H^q(\nabla) = 0$ *for* $q = 1, \ldots, m$ *(where* $n = 2m + 1$ *is the dimension of* M *). Then the product*

$$\widetilde{\tau}(M, \chi, g) \cdot V(M)^{-dimH^0(\nabla)}$$

is independent of the choice of metric g *, i.e. the metric-dependence of the Ray-Singer torsion* $\widetilde{\tau}(M, \chi, g)$ *factors out as* $V(M)^{dimH^0(\nabla)}$ *.*

In the special case where $dimM = 3$, $H^1(M) = 0$ and χ is the trivial representation the metric-independence of (4.14), i.e. of $\widetilde{\tau}(M, g) \cdot V(M)^{-1}$, was pointed out by Schwarz in [25].

(ii) We now show how the dependence of $\eta = \eta(0 \, | \, T_\nabla)$ on the connection map ∇ can be expressed via the formula [5, (I)] for the index of the twisted signature operator for a certain vectorbundle over $M \times [0, 1]$. Fix an arbitrary flat connection map $\nabla^{(0)}$ on $\Omega(M, \xi)$, define $\nabla^{(t)} = \nabla^{(0)} + t(\nabla - \nabla^{(0)})$, $t \in [0, 1]$, let $\widetilde{\xi}$ denote the pull-back of ξ to a vectorbundle over $M \times [0, 1]$ and let $\widetilde{\nabla}$ denote the connection map on $\Omega(M \times [0, 1], \widetilde{\xi})$ determined by $\nabla^{(t)}$. Pick a metric on M and equip $[0, 1]$ with the natural metric and orientation; these induce the product metric and orientation on $M \times [0, 1]$. Then, as in [5, (I),§4] we have the twisted signature map

$$(\widetilde{\nabla} + \widetilde{\nabla}^*)_+ \; : \; \Omega_+(M \times [0, 1], \xi \otimes \mathbf{C}) \longrightarrow \Omega_-(M \times [0, 1], \xi \otimes \mathbf{C}) \, .$$

With the Atiyah-Patodi-Singer boundary conditions the index of this map can be calculated from [5, (I),theorem(3·10)] to be

$$I_+(\nabla, \nabla^{(0)}) \;\; = \;\; \sum_{4j+2k=n+1} 2^k \int_M Lj(TM) \wedge Tch_k(\nabla, \nabla^{(0)}) - \eta(0 \, | \, T_\nabla)$$

$$+ \eta(0 \, | \, T_{\nabla^{(0)}}) - \sum_{p=0}^m \dim H^p(\nabla) - \sum_{q=0}^m \dim H^q(\nabla^{(0)}) \, .$$

$$(4.15)$$

Here $L_j(TM)$ is the j'th term in the Hirzebruch $L-$polynomial and

$$Tch_k(\nabla, \nabla^{(0)}) = \int_0^1 (\frac{i}{2\pi})^k \frac{1}{(k-1)!} Tr\Big((\nabla - \nabla^{(0)})\Omega_t^{k-1}\Big) dt$$

is the transgression of the k'th Chern character (with Ω_t the curvature tensor of $\nabla^{(t)}$). In deriving (4.15) we have used the following formulae for the

44

boundary operators $(-1)^l B^{(l)}$ of $(\widetilde{\nabla} + \widetilde{\nabla}^*)_+$, $l = 0,1$, where $B^{(0)}$ is the boundary operator at the boundary $M \times \{0\}$ of $M \times [0,1]$, and $-B^{(1)}$ is the boundary operator at $M \times \{1\}$. Setting $\nabla^{(1)} \equiv \nabla$ the $B^{(l)}$ are elliptic selfadjoint maps on $\Omega(M, \xi)$ defined on $q-$forms by[13]

$$B_q^{(l)} = (-i)^{\lambda(q)} \left(*\nabla^{(l)} + (-1)^{q+1}\nabla^{(l)}*\right) \qquad (4.16)$$

where $\lambda(q) = (q+1)(q+2) + m + 1$. Note that $B^{(l)}$ maps even forms to even forms and odd forms to odd forms and can therefore be decomposed as $B^{(l)} = B_{ev}^{(l)} \oplus B_{odd}^{(l)}$. As pointed out in [5, (I),§4] $B_{ev}^{(l)}$ is isomorphic to $B_{odd}^{(l)}$, so

$$\eta(s \,|\, B_{ev}^{(l)}) = \eta(s \,|\, B_{odd}^{(l)}) = \frac{1}{2}\eta(s \,|\, B^{(l)}) \,. \qquad (4.17)$$

The eta-functions of $B^{(l)}$ and $T_{\nabla^{(l)}}$ are now related by combining (4.17) with the following formulae:

$$\eta(s \,|\, B_{odd}^{(l)}) = \eta(s \,|\, * \nabla_m^{(l)}) \quad \text{for } m \text{ odd} \qquad (4.18)$$
$$\eta(s \,|\, B_{ev}^{(l)}) = \eta(s \,|\, i * \nabla_m^{(l)}) \quad \text{for } m \text{ even} \qquad (4.19)$$

These are analogous to [5, (I),proposition(4·20)] and can be derived by arguments analogous to the one given there. (They can also be derived by arguments similar to the one used in the proof of theorem 3.3 above). Combining (4.18), (4.19) and (4.17) with the expressions (4.2) and (4.10) for T_∇ we see that for m even or odd,

$$\eta(s \,|\, B^{(l)}) = 2\eta(s \,|\, T_{\nabla^{(l)}}) \,. \qquad (4.20)$$

Finally, in deriving (4.15) we have used the fact that $(B^{(l)})^2 = \oplus_{q=0}^n \Delta_q^{(l)}$ so that from Hodge-theory,

$$dim \, ker \, B^{(l)} = \sum_{q=0}^n dim H^q(\nabla^{(l)}) \,.$$

Taking the index formula (4.15) as an equation for $\eta = \eta(0 \,|\, T_\nabla)$ we see that the metric-dependence of η enters through $L_j(TM)$ and $\eta(0 \,|\, T_{\nabla^{(0)}})$. If

---

[13]The expression for the boundary operator depends on the convention used to define the Hodge star-map $*$. To get (4.16) we use $*\alpha \wedge \beta = \langle \alpha, \beta \rangle vol$ (rather than $\beta \wedge * \alpha = \langle \beta, \alpha \rangle vol$). Then (4.16) is in agreement with the expression [5, (I),(4·6)] derived in the case where m is odd.

$n = 3$ then the only contribution of the L-polynomial to (4.15) is $L_0 = 1$ and the metric-dependence of η is determined alone by $\eta(0\,|\,T_{\nabla^{(0)}})$. Defining $B^{(t)}$ by replacing $\nabla^{(l)}$ by $\nabla^{(t)}$ in (4.16) the index $I_+(\nabla, \nabla^{(0)})$ is the spectral flow of $B^{(t)}$ (or equivalently, twice the spectral flow of $T_{\nabla^{(t)}}$) as t runs from 0 to 1, see [5, (III),§§7–8].

4.2   The partition functions in the semiclassical approximation for the Witten-invariant

We take the gauge group to be $G = SU(N)$ and define an inner product in the Lie algebra $\mathbf{g}$ by $\langle a, b \rangle_{\mathbf{g}} = -\lambda_{\mathbf{g}}\mathrm{Tr}(ab)$ with scaling parameter $\lambda_{\mathbf{g}} > 0$. ($SU(N)$ and its Lie algebra are identified with their fundamental representations). The partition functions in the semiclassical approximation (1.14) are partition functions of functionals of the form $i\lambda S_{A_f}$, with $S_{A_f} = S_{\nabla}$ given by (4.1) with $m = 1$, $\xi = M \times \mathbf{g}$ with Hermitian structure determined by $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ and $\nabla = d_{A_f} = d + ad(A_f)$ (where A_f is a flat gauge field and $ad: \mathbf{g} \to End(\mathbf{g})$ is the adjoint representation, so $d_{A_f}\omega = d\omega + [A_f, \omega]$), and $\lambda = \frac{k}{4\pi\lambda_{\mathbf{g}}}$. These partition functions can be evaluated by the method described in this paper; they are given by (4.9) to be (for $k > 0$):

$$(2\pi\sqrt{\lambda_{\mathbf{g}}})^{\zeta(A_f)}\, e^{-\frac{i\pi}{4}\eta(A_f, g)}\, k^{-\zeta(A_f)/2}\, \tau(M, A_f, \langle \cdot, \cdot \rangle_{H(A_f)})^{1/2} \qquad (4.21)$$

where, from (4.12)

$$\zeta(A_f) = dimH^0(A_f) - dimH^1(A_f) \qquad (4.22)$$

and from (4.2)
$$\eta(A_f, g) = \eta(0\,|\,T_{A_f}) = \eta(0\,|\, * d_{A_f(1)}) \qquad (4.23)$$

(this depends on a metric g on M through the Hodge star-map $*$). We have substituted A_f in the notation in place of $\nabla = d_{A_f}$ and in place of the representation χ of $\pi_1(M)$ corresponding to d_{A_f}. The partition function (4.21) depends on a choice of inner products $\langle \cdot, \cdot \rangle_{H^0(A_f)}$ and $\langle \cdot, \cdot \rangle_{H^1(A_f)}$ in $H^0(A_f)$ and $H^1(A_f)$ respectively, which we denote collectively by $\langle \cdot, \cdot \rangle_{H(A_f)}$. From (4.13) we see that given a choice of orientation in $H^0(A_f)$ and $H^1(A_f)$ then as a functional of $\langle \cdot, \cdot \rangle_{H(A_f)}$ we can interpret (4.21) as an element in $\Lambda^{max}H^0(A_f) \otimes \Lambda^{max}H^1(A_f)^*$. Since the inner products $\langle \cdot, \cdot \rangle_q$ in the $\Omega^q(M, \mathbf{g})$ are all proportional to $\lambda_{\mathbf{g}}$ the adjoint maps ∇_q^* are independent of $\lambda_{\mathbf{g}}$. Therefore the dependence of $\tau(M, A_f, \langle \cdot, \cdot \rangle_{H(A_f)})^{1/2}$ on $\lambda_{\mathbf{g}}$ can only enter through

the adjoint maps ϕ_q^* which appear in the factor $det(\phi_0^*\phi_0)^{1/2}det(\phi_1^*\phi_1)^{-1/2}$ (c.f. (4.6) and (4.8)). If the inner products $\langle\,\cdot\,,\cdot\,\rangle_{H(A_f)}$ are chosen independent of $\lambda_{\mathbf{g}}$ then the partition function (4.21) is independent of $\lambda_{\mathbf{g}}$: In this case since $\langle\,\cdot\,,\cdot\,\rangle_q \sim \lambda_{\mathbf{g}}$ we have $\phi_q^* \sim \lambda_{\mathbf{g}}^{-1}$ so

$$det(\phi_0^*\phi_0)^{1/2}det(\phi_1^*\phi_1)^{-1/2} \sim \lambda_{\mathbf{g}}^{(-\frac{1}{2}dimH^0(A_f)+\frac{1}{2}dimH^1(A_f))}$$

and this cancels the factor $\sqrt{\lambda_{\mathbf{g}}}^{\zeta(A_f)}$ in (4.21). However, the canonical inner product in $H^0(A_f)$ described in the previous subsection is proportional to $\lambda_{\mathbf{g}}$ in the present case. Using it to define $\langle\,\cdot\,,\cdot\,\rangle_{H(A_f)}$ in cases where $H^1(A_f) = 0$ leads to $\tau(M, A_f, \langle\,\cdot\,,\cdot\,\rangle_{H(A_f)})$ being independent of $\lambda_{\mathbf{g}}$, so the dependence of the partition function (4.21) on $\lambda_{\mathbf{g}}$ in this case is given by $\sqrt{\lambda_{\mathbf{g}}}^{\zeta(A_f)} = (\lambda_{\mathbf{g}})^{\frac{1}{2}dimH^0(A_f)}$.

We now compare (4.21) with the expressions previously calculated by E. Witten and conjectured by D. Freed and R. Gompf, and by L. Jeffrey. We will not include the geometric counterterm, which, as discussed in the introduction, must be put in by hand to cancel the metric-dependence of the phase. The partition function of S_{A_f} with $\beta = i$ was evaluated by Witten [27, §2] in the case where the cohomology of d_{A_f} vanishes. This was done using standard methods of quantum field theory, i.e. gauge fixing implemented via a Lagrange-multiplier field. The method does not apply when the cohomology is non-vanishing. The expression obtained was

$$e^{-\frac{i\pi}{4}\eta_w(A_f,g)} \widetilde{\tau}(M, A_f)^{1/2} \tag{4.24}$$

where $\widetilde{\tau}$ is the usual Ray-Singer torsion of A_f and $\eta_w(A_f,g) = \eta(0\,|\,B^{(1)}_{odd})$ with $B^{(1)}_{odd}$ the restriction of $B^{(1)}$ in (4.16) to odd-degree forms with $m = 1$ and $\nabla^{(1)} = d_{A_f}$. From (4.17), (4.18) and (4.20) we have $\eta_w(A_f, g) = \eta(A_f, g)$, so (4.24) is identical to (4.21) when the cohomology of d_{A_f} vanishes.

Based on (4.24) Freed and Gompf [12, §1] conjectured an expression for the partition function of the functional $i\frac{k}{4\pi}S_{A_f}$ appearing in the semiclassical approximation (1.14) for the general case where the cohomology of d_{A_f} is not required to vanish. Some refinements were added by L. Jeffrey in [16, §5] leading to the following expression:

$$\frac{1}{|C(G)|}e^{-\frac{i\pi}{4}\eta(A_f,g)} k^{-\frac{1}{2}(dimH^0(A_f)-dimH^1(A_f))} < \tau^{1/2}(M, A_f), v_0 >_0 . \tag{4.25}$$

47

The motivation for the new features of this expression relative to (4.24) is as follows: In [12] the $k-$dependence of the Witten-invariant in the large$-k$ limit was calculated numerically for a number of lens spaces and Brieskorn spheres and found to be given by

$$k^{A_f^{max}\{-\frac{1}{2}dimH^0(A_f)+\frac{1}{2}dimH^1(A_f)\}} \qquad (4.26)$$

This led Freed and Gompf to conjecture the $k-$dependence

$$k^{-\frac{1}{2}(dimH^0(A_f)-dimH^1(A_f))} \qquad (4.27)$$

in $(4.25)^{14}$.

The phase $e^{-\frac{i\pi}{4}\eta(A_f,g)}$ in (4.25) was also shown numerically in [12] to be in agreement with the phase of the Witten-invariant in the large$-k$ limit (provided the geometric counterterms are included). The phase factor $\eta(A_f,g)$ was expressed in [12, (1.31)] via the index formula for the self-dual complex over $M \times [0,1]$ determined by d_{A_f}; it is also obtained as a special case of (4.15) with $\nabla^{(0)} = d$. (To see this note that the index $I_{sd}(A_f)$ of the self-dual complex [12, (1.30)] is half $I_+(d_{A_f},d)$, and for $G = SU(2)$ the transgression of the Chern character $Tch(d_{A_f},d)$ is $CS(ad(A_f)) = 4CS(A_f)$ (identifying $su(2)$ with its fundamental representation) where $CS(\cdot)$ is the Chern-Simons functional (1.13), and $dimH^q(d) = 3dimH^q(M)$).

The factor $< \tau^{1/2}(M,A_f),v_0 >_0$ in (4.25) is due to L. Jeffrey (in the case where $H^1(A_f)\neq 0$) and is defined as follows. As previously noted, given a choice of orientation of $H^0(A_f)$ and $H^1(A_f)$ we can consider $\tau^{1/2}$ as an element in $\Lambda^{max}H^0(A_f) \otimes \Lambda^{max}H^1(A_f)^*$. Given a volume form $v_0 \in \Lambda^{max}H^0(A_f)^*$, we obtain $< \tau^{1/2}(M,A_f),v_0 >_0 \in \Lambda^{max}H^1(A_f)^*$, where $< \cdot, \cdot >_0$ denotes the natural pairing of $\Lambda^{max}H^0(A_f)$ and $\Lambda^{max}H^0(A_f)^*$. In certain cases $H^1(A_f)$ can be identified with the tangentspace of the moduli space of flat gauge fields at A_f, then $< \tau^{1/2}(M,A_f),v_0 >_0$ is a top-degree form on this moduli space. It follows that for these cases the expression (1.14) for the semiclassical approximation can be generalised to the case where the moduli space of flat gauge fields is not discrete: the sum in (1.14) can be replaced by an integral over the moduli space. Note that if v_0 is determined by the inner product $\langle \cdot, \cdot \rangle_{H^0(A_f)}$ in $H^0(A_f)$ then $< \tau^{1/2}(M,A_f),v_0 >_0$ in (4.25) can be identified with $\tau(M,A_f,\langle \cdot, \cdot \rangle_{H(A_f)})^{1/2}$ in (4.21) considered as a functional

---

[14] According to [12] (4.27) was originally suggested by Witten in an informal lecture.

of the inner product $\langle\cdot\,,\cdot\rangle_{H^1(A_f)}$ in $H^1(A_f)$. Finally, $|C(G)|$ in (4.25) is the order of the center $C(G)$ of G; it is included to take account of the fact that constant gauge transformations corresponding to elements in $C(G)$ act trivially on gauge fields.

Comparing (4.21) with (4.25) we see that (4.21) considered as a functional of the inner product $\langle\cdot\,,\cdot\rangle_{H^1(A_f)}$ in $H^1(A_f)$ agrees with (4.25) up to a numerical factor $(2\pi)^{\zeta(A_f)}|C(G)|$. In particular, the phases and $k-$dependencies of (4.21) and (4.25) agree (although recall that we have omitted the geometric counterterm in the phase of the conjectured expression). As described in the introduction, for $G = SU(2)$ the conjectured formula (4.25) has been shown numerically [12] and analytically [16], [23] to be in agreement with the large$-k$ limit of the Witten-invariant for a wide class of 3-manifolds (lens spaces, Seifert manifolds and some Brieskorn spheres) up to minor numerical factors. (These typically involve a $\sqrt{2}$, although for S^3 a factor $\sqrt{\pi}$ is also involved). Thus, for these 3-manifolds, the semiclassical approximation with the partition functions calculated by the method described in this paper to give (4.21), with the geometric counterterms added by hand to the phase, agrees with the large$-k$ limit of the Witten-invariant modulo some numerical factors.

We now give an explanation for the discrepancy in the numerical factors between the semiclassical approximation and the large$-k$ limit of the Witten-invariant. It can be understood from recent work by the first author [1] combined with results of L. Rozansky [23]. In [1] a method for formally constructing the perturbative expansion of the formal expression (1.12) for the Witten-invariant about a flat gauge field A_f is described. It is based on a refinement of the Faddeev-Popov gauge-fixing procedure to take account of the fact that the Lorentz gauge-fixing condition $d^*_{A_f}\omega = 0$ does not determine ω uniquely, since, as is easily checked, the space of solutions ω is invariant under the subgroup $\mathcal{G}_{A_f}$ of gauge transformations which act trivially on A_f. The method requires $H^1(A_f) = 0$ but does not require $H^0(A_f) = 0$; it is inspired by, and extends the work of S. Axelrod and I. Singer [6] which requires $H^0(A_f) = H^1(A_f) = 0$. A generally assumed feature of perturbative QFT is that the term in the semiclassical approximation corresponding to a solution A_f of the classical field equations is the same as the lowest order term in a perturbative expansion about A_f. However, in [1] we find that the lowest order term differs from the term in the semiclassical approximation

49

by the inverse volume factor $V(\mathcal{G}_{A_f})^{-1}$. This factor also appeared in the work of Rozansky [23]: He invoked the normalisation procedure used to obtain physical quantities in QFT to argue that each partition function in the semiclassical approximation (1.14) should be modified by dividing by $V(\mathcal{G}_{A_f})$. (In other words, the factor $V(\mathcal{G}_{A_f})^{-1}$ is put in by hand in [23], whereas in [1] it is obtained by formal calculations carried out inside a self-contained framework). It is well-known that the isotropy groups $\mathcal{G}_{A_f}$ can be identified with subgroups of G, so the volumes $V(\mathcal{G}_{A_f})$ are finite.

Rozansky replaced the factor $|C(G)|$ in the conjectured expression (4.25) with the factor $V(\mathcal{G}_{A_f})^{-1}$, and invoked "an implicit factor $(\sqrt{2}\pi)^{-1}$ coming with each of the 1-dimensional integrals" in the integral (1.1) to obtain a modified conjectured expression for each partition function in the semiclassical approximation (1.14). Up to a power of $\sqrt{2}$ this is the same as multiplying our expression (4.21) by $V(\mathcal{G}_{A_f})^{-1}$. Rozansky showed that with this expression the semiclassical approximation agrees with the large$-k$ limit of the Witten-invariant in all the cases for which this limit has been analytically calculated (i.e. S^3, lens spaces, Seifert manifolds). It follows that up to a power of $\sqrt{2}$ the lowest order term in the sum over A_f of the perturbative expansions of (1.12) about all non-equivalent A_f, calculated by the method of [1], agrees with the large$-k$ limit of the Witten-invariant for these manifolds, as predicted by QFT. Complete agreement can be obtained by modifying the values of the volumes $V(\mathcal{G}_{A_f})^{-1}$ in [23] by powers of $\sqrt{2}$.[15] In particular this gives a self-contained method for deriving the factor π in the large$-k$ limit $\sqrt{2}\pi k^{-3/2}$ of the Witten-invariant for S^3, which has previously been a source of mystery[16]. We will demonstrate this explicitly below.

An intuitive explanation for why the partition functions in the semiclassical approximation (1.14), calculated by the method described in this paper, fail to produce the factors $V(\mathcal{G}_{A_f})^{-1}$ is the following: Loosely speaking, the semiclassical approximation (1.14) does not know about $\mathcal{G}_{A_f}$, it only knows about its infinitessimal version, i.e. its Lie algebra, which is $H^0(A_f)$. Therefore the factors $V(\mathcal{G}_{A_f})^{-1}$ in the perturbative expansions are replaced by the divergent factors $V(H^0(A_f))^{-1}$ in the semiclassical approximation, which are divided out (i.e. discarded) when normalising the partition functions. In-

---

[15]There is sufficient arbitrariness in the "reasonable" values for $V(\mathcal{G}_{A_f})$ given in [23] to allow for this.

[16]See e.g. [12, p.113 problem (1)], [16, footnote p.587].

deed, the formal expression for each partition function includes divergent volume factors $V(\Gamma_k)^{\pm 1}$ and $V(H^k(R(S)))^{\pm 1}$ (see (2.23)); when $H^1(A_f) = 0$ the contribution to these volumes from the cohomology in the present case is precisely $V(H^1(R(S)))^{-1} = V(H^0(A_f))^{-1}$.

We now give an illustration of the points discussed above by explicitly calculating the semiclassical approximation (1.14) for the manifold S^3 with $G = SU(2)$. Up to gauge equivalence $A_f = 0$ is the only flat gauge field on S^3, so the semiclassical approximation obtained from our extension of Schwarz's method is given by (4.21) with $A_f = 0$. We include Witten's geometric counterterm; in the present case this cancels precisely the phase in (4.21). For reasons discussed above we include the inverse volume factor $V(\mathcal{G}_{A_f})^{-1}$. For $A_f = 0$ clearly $\mathcal{G}_{A_f} = G = SU(2)$, so this factor is $V(SU(2))^{-1}$. Then, since $H^0(A_f) = su(2)$ and $H^1(A_f) = 0$ we get from (4.21)

$$(2\pi)^3 \, \lambda_{\mathbf{g}}^{3/2} \, k^{-3/2} \, \tau(S^3, A_f=0, \langle \cdot, \cdot \rangle_{H^0(A_f=0)})^{1/2} \, V(SU(2))^{-1} \qquad (4.28)$$

where as before we take the inner product in $\mathbf{g} = su(2)$ to be $\langle a, b \rangle = -\lambda_{\mathbf{g}} Tr(ab)$. This gives an inner product in $H^0(A_f=0) \cong su(2)$ and determines $V(SU(2)) = (\frac{2}{\lambda_{\mathbf{g}}})^{3/2} 16\pi^2$. By the same calculations as those leading to (4.14) we find

$$\tau(S^3, A_f=0, \langle \cdot, \cdot \rangle_{H^0(A_f=0)})^{1/2} = \widetilde{\tau}(S^3)^{3/2} \, V(S^3)^{-3/2} \qquad (4.29)$$

where $\widetilde{\tau}(S^3)$ is the usual Ray-Singer torsion of S^3 and $V(S^3)$ is its volume. The standard embedding $S^3 \hookrightarrow \mathbf{R}^4$ induces a metric in S^3 from the standard metric in $\mathbf{R}^4$; using this we get $V(S^3) = 2\pi^2$. From the calculations in [19, §4] we get $\widetilde{\tau}(S^3) = e^{-4\zeta_R'(0)}$, where $\zeta_R(s)$ is the analytic continuation of the Riemann zeta-function. From [7, p.26] $\zeta_R'(0) = -\frac{1}{2}\log(2\pi)$, so $\widetilde{\tau}(S^3) = (2\pi)^2$. Substituting these expressions into (4.28) gives

$$\frac{(2\pi)^3 \lambda_{\mathbf{g}}^{3/2} k^{-3/2} \widetilde{\tau}(S^3)^{3/2}}{V(S^3)^{3/2} V(SU(2))} = \frac{(2\pi)^3 \lambda_{\mathbf{g}}^{3/2} k^{-3/2} (2\pi)^3}{(2\pi^2)^{3/2} (\frac{2}{\lambda_{\mathbf{g}}})^{3/2} 16\pi^2} = \sqrt{2}\pi k^{-3/2} \left(\frac{\lambda_{\mathbf{g}}}{\sqrt{2}}\right)^3. \quad (4.30)$$

QFT predicts that this should coincide with the Witten-invariant $Z_W(k)$ in the limit of large k. The calculation of the Witten-invariant for S^3 [27, §4] involves very different mathematics to that used to obtain (4.30): S^3 is obtained by surgery on $S^2 \times S^1$ and $Z_W(k)$ for S^3 turns out to be a matrix

element of the representation of the corresponding diffeomorphism of the torus on the characters of the irreducible level$-k$ representations of the loop group of G. For $G = SU(2)$ this is

$$Z_W(k) = \sqrt{\frac{2}{k+2}} \, \sin\left(\frac{\pi}{k+2}\right) \ \sim \ \sqrt{2}\pi k^{-3/2} \quad \text{for } k \to \infty. \qquad (4.31)$$

Thus (4.30) reproduces the factor π in the large$-k$ limit (4.31). To reproduce the correct power of $\sqrt{2}$ in (4.30) we must set $\lambda_{\mathbf{g}} = \sqrt{2}$, a somewhat peculiar value.

Appendix

For $A = (A_N, \ldots, A_1, A_0) \in GL(\Gamma_N) \times \cdots \times GL(\Gamma_1) \times GL(\Gamma)$ define $T_k^{*(A)}$ to be the adjoint of $T_k : \Gamma_k \longrightarrow \Gamma_{k-1}$ determined by the inner products $\langle \cdot, \cdot \rangle_k^{A_k}$ and $\langle \cdot, \cdot \rangle_{k-1}^{A_{k-1}}$. Define $W^{\perp(A)}$ to be the orthogonal complement of a subspace $W \subseteq \Gamma_k$ determined by $\langle \cdot, \cdot \rangle_k^{A_k}$; it is easy to see that $W^{\perp(A)} = (A_k^* A_k)^{-1}(W^\perp)$. In the following we will use the relations $T_k = T_k(P_k + Q_k) = \tilde{T} Q_k$ and, when the cohomology of the resolvent vanishes, $T_k = (P_{k-1} + Q_{k-1})T_k = P_{k-1}T_k$. Recall that the cohomology of the resolvent is assumed to vanish for (2.13) and (2.14), but not for (2.31).

Proof of formula (2.13).
We calculate $det(\widetilde{T}_{A_0})$, where $\widetilde{T}_{A_0}$ is the restriction of $T_{A_0} = (A_0^* A_0)^{-1}T$ to $ker(T_{A_0})^{\perp(A)} = ker(T)^{\perp(A)} = (A_0^* A_0)^{-1}(ker(T)^\perp)$:

$$
\begin{aligned}
det(\widetilde{T}_{A_0}) &= det\left((A_0^* A_0)^{-1}T\Big|_{(A_0^* A_0)^{-1}(ker(T)^\perp)}\right) \\
&= det\left(T(A_0^* A_0)^{-1}\Big|_{Im(T)=ker(T)^\perp}\right) \\
&= det\left(\tilde{T} Q_0 (A_0^* A_0)^{-1} Q_0\right) \\
&= det(\tilde{T}) \, det(Q_0(A_0^* A_0)^{-1} Q_0)
\end{aligned}
$$

proving formula (2.13).

Proof of formula (2.14).
We calculate $det((\widetilde{T}_{k(A)})^{*(A)}\widetilde{T}_{k(A)})$, where $\widetilde{T}_{k(A)}$ is the restriction of T_k to $ker(T_k)^{\perp(A)} = (A_k^* A_k)^{-1}(ker(T_k))$. For $v \in \Gamma_k$, $w \in \Gamma_{k-1}$ we have

$$\langle w, T_k v\rangle_{k-1}^{A_{k-1}} = \langle (A_k^* A_k)^{-1}T_k^*(A_{k-1}^* A_{k-1})w, v\rangle_k^{A_k}$$

so $T_k^{*(A)} = (A_k^* A_k)^{-1} T_k^* (A_{k-1}^* A_{k-1})$. It follows that

$$
\begin{aligned}
det(\widetilde{T}_{k(A)}^* \widetilde{T}_{k(A)}) &= det\left(T_k^{*(A)} T_k\Big|_{(A_k^* A_k)^{-1}(ker(T_k)^\perp)}\right) \\
&= det\left((A_k^* A_k)^{-1} T_k^*(A_{k-1}^* A_{k-1}) T_k\Big|_{(A_k^* A_k)^{-1}(ker(T_k)^\perp)}\right) \\
&= det\left(T_k (A_k^* A_k)^{-1} T_k^*(A_{k-1}^* A_{k-1})\Big|_{Im(T_k)}\right) \\
&= det\left(\widetilde{T}_k Q_k (A_k^* A_k)^{-1} Q_k \widetilde{T}_k^* P_{k-1}(A_{k-1}^* A_{k-1}) P_{k-1}\right) \\
&= det(\widetilde{T}_k \widetilde{T}_k^*) \, det\left(Q_k (A_k^* A_k)^{-1} Q_k\right) \\
&\quad \times det\left(P_{k-1}(A_{k-1}^* A_{k-1}) P_{k-1}\right)
\end{aligned}
$$

proving formula (2.14).

   Proof of formula (2.31).
Define $\mathcal{H}_{k(A)}$ and $\Phi_{k(A)} \colon \mathcal{H}_{k(A)} \xrightarrow{\simeq} H^k(R(S))$ as in (2.18) and (2.19) with the inner product $\langle \cdot, \cdot \rangle_k$ in Γ_k replaced by $\langle \cdot, \cdot \rangle_k^{A_k}$. Abbreviating $H^k(R(S))$ by H^k we define maps for $k = 1, \ldots, N$:

$$
\begin{aligned}
\mathcal{T}_k &= T_k \oplus \Phi_{k-1}^{-1} \colon \Gamma_k \oplus H^{k-1} \longrightarrow Im(T_k) \oplus \mathcal{H}_{k-1} = ker(T_{k-1}) \\
\mathcal{T}_{k(A)} &= T_k \oplus \Phi_{k-1(A)}^{-1} \colon \Gamma_k \oplus H^{k-1} \longrightarrow Im(T_k) \oplus_{A_{k-1}} \mathcal{H}_{k-1(A)} = ker(T_{k-1})
\end{aligned}
$$

where $\oplus_{A_{k-1}}$ denotes direct sum orthogonal w.r.t. $\langle \cdot, \cdot \rangle_{k-1}^{A_{k-1}}$. Then the formula (2.31) is equivalent to

$$
\begin{aligned}
det&\left(\mathcal{T}_{k(A)}^{*(A)} \mathcal{T}_{k(A)}\Big|_{ker(T_k)^{\perp(A)} \oplus H^{k-1}}\right) \\
&= det\left(Q_k (A_k^* A_k)^{-1} Q_k\right) det\left(P_{k-1}(A_{k-1}^* A_{k-1}) P_{k-1}\right) \\
&\quad \times det\left(\mathcal{T}_k^* \mathcal{T}_k\Big|_{ker(T_k)^\perp \oplus H^{k-1}}\right) \qquad\qquad (A.1)
\end{aligned}
$$

where the adjoint map $\mathcal{T}_{k(A)}^{*(A)}$ is defined w.r.t. the inner products $\langle \cdot, \cdot \rangle_k^{A_k} \oplus \langle \cdot, \cdot \rangle_{H^{k-1}}$ in $\Gamma_k \oplus H^{k-1}$ and $\langle \cdot, \cdot \rangle_{k-1}^{A_{k-1}}$ in Γ_{k-1}. An argument analogous to the preceding proof of (2.14) gives

$$
\begin{aligned}
det&\left(\mathcal{T}_{k(A)}^{*(A)} \mathcal{T}_{k(A)}\Big|_{ker(T_k)^{\perp(A)} \oplus H^{k-1}}\right) \\
&= det\left(Q_k (A_k^* A_k)^{-1} Q_k\right) det\left(P_{k-1}(A_{k-1}^* A_{k-1}) P_{k-1}\right) \\
&\quad \times det\left(\mathcal{T}_{k(A)}^* \mathcal{T}_{k(A)}\Big|_{ker(T_k)^\perp \oplus H^{k-1}}\right) \qquad\qquad (A.2)
\end{aligned}
$$

It is clear from the definitions that $P_{[\mathcal{H}_{k-1}]}\Phi^{-1}_{k-1(A)} = \Phi^{-1}_{k-1}$, where $P_{[\mathcal{H}_{k-1}]}$ is the orthogonal projection map of Γ_{k-1} onto $\mathcal{H}_{k-1}$. It follows that the restriction of $\mathcal{T}_{k(A)}$ to $ker(T_k)^{\perp} \oplus H^{k-1}$ has the form

$$\begin{pmatrix} \widetilde{T}_k & Q_k \Phi^{-1}_{k-1(A)} \\ 0 & \Phi^{-1}_{k-1} \end{pmatrix} : ker(T_k)^{\perp} \oplus H^{k-1} \xrightarrow{\simeq} Im(T_k) \oplus \mathcal{H}_{k-1}$$

so

$$\begin{aligned} det\left(\mathcal{T}^*_{k(A)}\mathcal{T}_{k(A)}\Big|_{ker(T_k)\oplus H^{k-1}}\right) &= det(\widetilde{T}^*_k\widetilde{T}_k)\,det((\Phi^{-1}_{k-1})^*\Phi^{-1}_{k-1}) \\ &= det\left(\mathcal{T}^*_k\mathcal{T}_k\Big|_{ker(T_k)^{\perp}\oplus H^{k-1}}\right). \end{aligned}$$

Substituting this into (A.2) gives (A.1), proving formula (2.31).

References

[1] Adams, D.H.: Perturbative expansion in quantum theories for gauge fields on manifolds. (In preparation); Chern-Simons perturbation theory for S^3. Talk given at the First Irish Quantum Field Theory Conference, Trinity College, Dublin, May 1994. (Copies available from the author).

[2] Aitken, A.C.: Determinants and matrices. (Fourth edition). Oliver and Boyd, Edinburgh and London (1946)

[3] Atiyah, M.F.: Topological quantum field theory. Publ. Math. Inst. Hautes Etudes Sci. (Paris) **68**, 175–186 (1989)

[4] Atiyah, M.F.: The geometry and physics of knots. Cambridge University Press, (1990)

[5] Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry. I. Math. Proc. Cambridge Philos. Soc. **77**, 43–69 (1975); II. Math. Proc. Cambridge Philos. Soc. **78**, 405–432 (1975); III. Math. Proc. Cambridge Philos. Soc. **79**, 71–99 (1976)

[6] Axelrod, S., Singer, I.M.: Chern-Simons perturbation theory. I. Proc. XX'th DGM Conf. (S. Catto, A. Rocha, eds), World Scientific, Singapore and Teaneck, NJ, 1992, pp3–45; II. J. Differential Geom. **39** 173–213 (1994)

[7] Bateman manuscript project: Higher transendental functions, vol 1. McGraw-Hill, New York (1953)

[8] Birmingham, D., Blau, M., Rakowski, M., Thompson, G.: Topological field theory. Phys. Rep. **209**, 129–340 (1991)

[9] Blau, M., Thompson, G.: Localisation and Diagonalisation: A review of functional integral techniques for low-dimensional gauge theories and topological field theories. ICTP preprint IC–95–5, hep-th/9501075. (To appear in J. Math. Phys. Special Issue on Functional Integration, May 1995).

[10] Dijkgraaf, R., Witten, E.: Topological gauge theory and group cohomology. Commun. Math. Phys. **129**, 393–429 (1990)

[11] Freed, D.S.: Classical Chern-Simons theory I. Preprint, Maths dept., University of Texas at Austin. To appear in Adv. Math.

[12] Freed, D.S., Gompf, R.E.: Computer calculation of Witten's 3-manifold invariant. Commun. Math. Phys. **141**, 79–117 (1991)

[13] Freed, D.S., Gompf, R.E.: Computer tests of Witten's Chern-Simons theory against the theory of three-manifolds. Phys. Rev. Lett. **66**, 1255–1258 (1991)

[14] Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: Discriminants, resultants and multidimensional determinants. Birkhauser, Boston (1994)

[15] Gilkey, P.B.: Invariance theory, the heat equation and the Atiyah-Singer index theorem. Publish or Perish, Delaware (1984)

[16] Jeffrey, L.C.: Chern-Simons-Witten invariants of lens spaces and torus bundles, and the semiclassical approximation. Commun. Math. Phys. **147**, 563–604 (1992)

[17] Kirby, R., Melvin, P.: The three-manifold invariants of Witten and Reshetikhin-Turaev. Invent. Math. **105**, 473–545 (1991)

[18] Rama, S.K., Sen, S.: Comments on Witten invariants of three-manifolds for $SU(2)$ and $\mathbf{Z}_m$. Mod. Phys. Lett. A, **8**, 2285–2297 (1993)

[19] Ray, D.B.: Reidemeister torsion and the Laplacian on lens spaces. Adv. Math. **4**, 109–126 (1970)

[20] Ray, D.B., Singer, I.M.: R-torsion and the Laplacian on Riemannian manifolds. Adv. Math. **7**, 145–210 (1971)

[21] Ray, D.B., Singer, I.M.: Analytic torsion. Partial Differential Equations, Proceedings of Symposia in Pure Mathematics, Vol.23, pp 167–182. American Mathematical Society 1973

[22] Reshetikhin, N., Turaev, V.: Ribbon graphs and their invariants derived from quantum groups. Commun. Math. Phys. **127**, 1–26 (1990); Invariants of three-manifolds via link polynomials and quantum groups. Invent. Math. **103** 547–597 (1991)

[23] Rozansky, L.: A large k asymptotics of Witten's invariant of Seifert manifolds. Preprint UTTG–06–93, Dept. of Physics, University of Texas at Austin.

[24] Schwarz, A.S.: The partition function of degenerate quadratic functional and Ray-Singer invariants. Lett. Math. Phys. **2**, 247–252 (1978)

[25] Schwarz, A.S.: The partition function of a degenerate functional. Commun. Math. Phys. **67**, 1–16 (1979)

[26] Walker, K.: On Witten's three-manifold invariants. Preprint.

[27] Witten, E.: Quantum field theory and the Jones polynomial. Commun. Math. Phys. **121**, 351–399 (1989)